Complete zeta-function approach to the electromagnetic Casimir effect for spheres and circles.

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Abstract

A technique for evaluating the electromagnetic Casimir energy in situations involving spherical or circular boundaries is presented. Zeta function regularization is unambiguously used from the start and the properties of Bessel and related zeta functions are central. Nontrivial results concerning these functions are given. While part of their application agrees with previous knowledge, new results like the zeta-regularized electromagnetic Casimir energy for a circular wire are included.

1 Introduction

Casimir effect problems have been showing a remarkably long-lived appeal since the day of their birth[1], still stirring up intense activity all along the eighties ([2]-[6], to name just a few works), and reaching present-day topics (e.g. [7]-[11]). During all this time they have been object of many different approaches: stress-energy tensor [12], Green function methods [13], multiple scattering expansions [14], heat-kernel series ([15]-[19]), etc.

In the present paper, we offer new calculations of the Casimir energy for an e.m. field in the presence of a sphere in $D = 3$, and of a circle in $D = 2$, with perfect-conductor conditions at their points. The contributions from inside and outside the boundary are separately studied. Further, in $D = 3$ we take apart the pieces associated to trasnverse electric (TE) and transverse magnetic (TM) modes in each case. Although this is lengthier than starting from the whole sum (since some pieces which cancel would then be eliminated at the beginning, and now we keep them until the end) we get as a bonus a useful decomposition into the four contributions. Thus, we find as a byproduct the effect coming from e.g. the interior part only or the Dirichlet modes by themselves.

Unlike previous works on this subject involving zeta-functions at some stage or other (e.g. [20, 10]), we adopt here what might be called a 'straight' or 'frontal' zeta-regularization approach right at the outset from the eigenmode summations themselves, as advocated e.g. in [6]

We believe that our attitude is quite sound, as generic zeta function regularization [21, 22] is already a widespread technique. This method, often applied when boundary conditions affect a given quantum system

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—like quantum billiards [23, 24]— or field theory, is also adequate for dealing with finite temperature or curved spaces. Moreover, its variants have suceeded in problems involving nontrivial topologies or boundaries, like Kaluza-Klein style models [25, 26], or systems under the influence of external fields [27] or in interaction with material bodies (e.g. superconductors [28]) —other applications are shown in [29].

Many of these advances came true by means of a precise knowledge of the spectral zeta function associated to each system, which is the key to the derivation of many results. Somehow, this happens also to the present paper: our calculation is indebted to the investigation started in [30, 31] about Bessel zeta functions[33, 34], and taken further in [38] for the case of a purely scalar field inside a sphere, under Dirichlet boundary conditions. That work is here extended so as to include all the field modes appearing in the problems under discussion. It is also worth to note that a similar technique has been applied in Quantum Cosmology to study the Hartle-Hawking wave function of the universe [39].

After a brief survey on electromagnetism (sect. 2) general considerations about the physical problem of the Maxwell modes, and its spectral zeta function are made in sect. 3. First, we study in some detail the part associated to the internal transverse electric (TE) modes with particular emphasis on the zeta function for the zeros of the J_ν Bessel function and on the construction of the corresponding complete zeta function. Then, the same procedure is applied to the internal transverse magnetic (TM) part and its analogous zeta function. Next, the method is repeated for external modes in sect. 4. Comments about the result in these and related situations are made in sect. 5. Essential mathematical material concerning the derivation of partial wave zeta functions for external modes has been packaged into app. A. App. B contains an example of complete zeta function alternative calculation by using the coefficients of the heat-kernel series for the Laplacian. A comparison with previous estimates for $D = 2$ is made in app. C.

2 Electromagnetism in $D = 3, 2$

2.1 Neutral and perfectly conducting sphere in $D = 3$

We shall briefly sketch the classical problem of an electromagnetic (e.m.) field kept within a cavity resonator bounded by a perfectly conducting spherical shell of radius a. The adequate conditions on the surface are $\vec{n} \cdot \vec{B}|_{r=a} = \vec{n} \times \vec{E}|_{r=a} = 0$ (where $\vec{n}$ is the normal vector), in addition to the requirement of regularity in the interior. Then, the spherical e.m. waves which result from solving Maxwell's equations have radial parts $\propto r^{1-D/2} J_{\nu(D,l)}(\omega r)$, where

$$\nu(D,l) = l + \frac{D}{2} - 1 \tag{2.1}$$

denotes the Bessel index for angular monentum l in $D = 3$ space dimensions. In the present cirsumstances, the solutions are divided into 'transverse electric' (TE) and 'transverse magnetic' (TM) ones (see e.g. [40, 41]). Their possible associated frequencies (ω) are then determined by conditions on the radial parts which read, in each case,

$$\left. r^{1-D/2} J_{\nu(D,l)}(\omega r) \right|_{r=a} = 0, \text{ for region I TE-modes,} \tag{2.2}$$

$$\left. \frac{d}{dr} \left(r^{D-2}\, r^{1-D/2} J_{\nu(D,l)}(\omega r) \right) \right|_{r=a} = 0, \text{ for region I TM-modes,} \tag{2.3}$$

Clearly, when a particular solution is of TE type, ωa has to be a nonvanishng zero of a Bessel function $J_{\nu(D,l)}$, for some l. (2.3) can also be written as

$$(D/2-1)J_{\nu(D,l)}(k)+kJ'_{\nu(D,l)}(k)=0,\ k\equiv\omega a, \tag{2.4}$$

which is a Robin (or standard homogeneous) condition with relative coefficients $(D/2-1,1)$. If one keeps in mind the Maxwell equations, all this is valid for $D=3$ only but, when regarded as just two massless scalar fields obeying the Klein-Gordon equation and subject to Dirichlet and (a specific) Robin b.c., it holds for general D[1]. The same boundary conditions apply to the external (region II) solutions and we shall come back later to this question.

What is more, taking advantage of the $D=3$ duality $\vec{E}\leftrightarrow\vec{B}$ between e.m. and colour gauge fields one realizes that, up to a global factor equal to the number of SU(N_c) degrees of freedom, the present set-up is equivalent to a bag model for linear QCD[2] including only internal gluon modes.

2.2 Neutral and perfectly conducting circular wire in $D=2$

Two-dimensional spaces can be of interest since they are the scenario for systems such as quantum billiards[34, 23, 24] or anyons [35]. In particular, the electromagnetic–Chern-Simons Casimir effect in a $(2+1)$-dimensional spacetime has already been studied for parallel conducting lines in [36] and for a circle in [37]. What is explained in this subsection might also be obtained by the formalism set up in sects I and II of [37] when the Chern-Simmons part is absent. The components of the e.m. tensor in a $(2+1)$-dimensional spacetime are specified in terms of its electric and magnetic fields

$$F_{\mu\nu}=\begin{pmatrix}0 & E_1 & E_2\\ -E_1 & 0 & B\\ -E_2 & -B & 0\end{pmatrix}.$$

In $D=2$ there is room for only one $\vec{B}$-component. Actually the magnetic field has become a scalar rather than a vector. We impose the Maxwell equations in the vacuum (absence of charges and current density)

$$\partial_\mu F^{\mu,\nu}=0 \tag{2.5}$$

$$\partial_\alpha F^{*\ \alpha}=0, \tag{2.6}$$

where $F^{*\ \alpha}=\frac{1}{2}\epsilon^{\mu\nu\alpha}F_{\mu\nu}$ is the dual tensor to the e.m. field. More explicitly we have $F^{*\ 0}=B$ $F^{*\ 1}=-E_2$ and $F^{*\ 2}=E_1$. From (2.6) we get

$$\text{rot } E=\dot{B}, \tag{2.7}$$

where we understand that rot $E\equiv\partial_1E_2-\partial_2E_1$. From equation (2.5) we draw

$$\begin{aligned}\text{div }\vec{E} &= 0 \quad (2.8)\\ \dot{E}_1+\partial_2B &= 0 \quad (2.9)\\ \dot{E}_2-\partial_1B &= 0. \quad (2.10)\end{aligned}$$

[1] The possible relevance of the space dimension as a perturbation parameter was noted in [42].

[2] by 'linear' we mean that the nonlinear $f^{abc}A^\mu_aA^\nu_b$ term is omitted from the field-strength tensor, as usual in such approaches.

(2.7)-(2.10) are the complete set of fundamental equations. Taking the time-derivative of (2.7), ∂_2 of (2.9), ∂_1 of (2.10) and combining the results, one finds $\nabla^2 B - \ddot{B} = 0$ .i.e.

$$\Box B = 0 \tag{2.11}$$

which shows that B obeys a Klein-Gordon equation. Now, we give the form of these expressions when the fields have a time dependence of the form $\vec{E}(\vec{x}, t) = \vec{\epsilon}(\vec{x})e^{-i\omega t}$ and $\vec{B}(\vec{x}, t) = \vec{b}(\vec{x})e^{-i\omega t}$

$$\begin{aligned} \text{rot } \vec{\epsilon} &= -i\omega b, && (2.12)\\ \text{div } \vec{\epsilon} &= 0, && (2.13)\\ i\omega\epsilon_1 - \partial_2 b &= 0, && (2.14)\\ i\omega\epsilon_2 + \partial_1 b &= 0. && (2.15) \end{aligned}$$

From this set of equations —or straighforwardly from (2.11) and the time dependence of $\vec{B}(\vec{x}, t)$— we get that $(\Delta + \omega^2)b = 0$. Once this equation is solved, the electric field is given by equations (2.14), (2.15). The ensuing solution may be easily seen to automatically satisfy expression (2.12). Now we arrive at the question of boundary conditions. Ancient lore tells us that the electric field must be orthogonal to the surface of a perfect conductor; if the normal vector is given by (n_1, n_2), this condition takes the form $0 = n_1 E_2 - n_2 E_1$, which, using (2.14), (2.15) is equivalent to $0 = \frac{\partial}{\partial \vec{n}} B$. To sum up: the problem is reduced to that of a scalar field B which satisfies the typical Helmholtz equation and satisfies Neumann boundary conditions. In our notations, for $D = 2$ one has $\alpha(2) = 0$ echoing the conversion of the Robin b.c. into a purely Neumann condition.

The reader may check that all the modes generate a non-zero total charge on the boundary (which is given by the scalar product of the electric field and the normal vector to the boundary).

3 Internal modes

Within these types of set-up, zero-point energies emerge from vacuum mode-sums of the type $\hbar c \dfrac{1}{2} \sum_n \omega_n$, and give rise to the celebrated Casimir effect [1, 4]. Note that the summation extends over all the ω_n's in the set of eigenmodes. As a result, sums of this sort do usually diverge and call for some regularization to make sense of them.

At this point, we introduce the usual spectral zeta functions, which we denote by

$$\zeta_{\mathcal{M}}(s) = \sum_n \omega_n^{-s}, \qquad \zeta_{\frac{\mathcal{M}}{\mu}}(s) = \sum_n \left(\frac{\omega_n}{\mu}\right)^{-s}, \tag{3.1}$$

μ is an arbitrary scale with mass dimensions, often used to work with dimensionless objects. As they stand, these identities hold only for Re $s > s_0$, being s_0 a positive value given by the rightmost pole of $\zeta_{\mathcal{M}}(s)$. However, such a function admits analytic continuation to other values of s, in particular, to negative reals.

The finite part of the vacuum energy —E_C— can be found by combining zeta-regularization of the mode-sum and a principal part prescription from ref.[6]. Following that work, one may put

$$E_C(\mu) = \mathrm{PP}_{s\to -1}\left[\frac{1}{2}\hbar c\,\mu\,\zeta_{\frac{\mathcal{M}}{\mu}}(s)\right], \tag{3.2}$$

where PP denotes principal part. One should be aware that the whole, observable, physical energy includes other terms which have to do with the couplings of the bag Lagrangian — see the discussion on this point in the same

reference. (From here on, we adopt the typical QFT units which make $\hbar = c = 1$). Obviously, for this procedure to work we must know how to obtain the analytic continuation of $\zeta_{\mathcal{M}}(s)$ to —at least— a small part of the negative real axis.

In order to proceed, we shall introduce 'partial-wave' zeta functions for fixed values of the Bessel index ν. We define the zeta function for the zeros of J_ν as (see also [30, 31]).

$$\zeta_\nu^{\mathrm{I},\mathcal{D}}(s) = \sum_{n=1}^{\infty} j_{\nu,n}^{-s}\,, \ \text{for Re } s > 1, \tag{3.3}$$

where $j_{\nu n}$ denotes the nth nonvanishing zero of J_ν. The $\mathrm{I},\mathcal{D}$ label has been added as a reminder that this comes from eigenmodes in region I dictated by Dirichlet-type b.c.[3] (Discrete versions of the Bessel problem, their solutions and associated zeta functions have also received some attention in [43]). Analogously, let

$$\zeta_\nu^{\mathrm{I},\mathcal{R}}(s) = \sum_{n=1}^{\infty} k_{\nu,n}^{-s}\,, \ \text{for Re } s > 1, \tag{3.4}$$

with $k_{\nu n}$ denoting the nth solution of eq. (2.4) for a given ν.

Reconsidering the same problem in D-dimensional space, taking into account the degeneracy $d(D,l)$ of each angular mode in D dimensions, we define the 'complete' spherical zeta functions

$$\begin{aligned}
\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{D}}(s) &= a^s \sum_{l=l_{\min}}^{\infty} d(D,l) \sum_{n=1}^{\infty} j_{\nu(D,l),n}^{-s} &= a^s \sum_{l=l_{\min}}^{\infty} d(D,l)\, \zeta_{\nu(D,l)}^{\mathrm{I},\mathcal{D}}(s), \\
\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{R}}(s) &= a^s \sum_{l=l_{\min}}^{\infty} d(D,l) \sum_{n=1}^{\infty} k_{\nu(D,l),n}^{-s} &= a^s \sum_{l=l_{\min}}^{\infty} d(D,l)\, \zeta_{\nu(D,l)}^{\mathrm{I},\mathcal{R}}(s).
\end{aligned} \tag{3.5}$$

$l_{\min}$ is the minimum value of l and, for gauge fields in $D = 3$, $l_{\min} = 1$. If we consider scalar fields, then $l_{\min} = 0$.

As for $d(D,l)$, we find its value in [44] and put

$$d(D,l) = (2l + D - 2)\frac{(l + D - 3)!}{l!(D-2)!} = \frac{2}{(D-2)!} \sum_{k=0}^{k_{\max}(D)} (-1)^k \mathcal{A}_k(D) \nu(D,l)^{D-2-2k}, \tag{3.6}$$

where

$$k_{\max}(D) = \begin{cases} \dfrac{D-3}{2} & \text{for odd } D \geq 3, \\ \dfrac{D}{2} - 2 & \text{for odd } D \geq 4, \end{cases} \tag{3.7}$$

and the form of the $\mathcal{A}_k(D)$'s can be read off from [45] (see also [38]).

3.1 Internal Dirichlet (TE) modes

3.1.1 'Partial-wave' zeta function

By (3.2), computing the Casimir energy calls for the knowledge of the Bessel zeta functions (3.3) at $s = -1$, while the complex domain where (3.3) holds is bounded by Re $s = 1$. Fortunately for us, $\zeta_\nu^{\mathrm{I},\mathcal{D}}(s)$ admits an analytic continuation to other values of s. What is more, in refs. [30] and [31] we showed how to obtain an integral representation valid for $-1 <$ Re $s < 0$, which reads

$$\zeta_\nu^{\mathrm{I},\mathcal{D}}(s) = \frac{s}{\pi} \sin\frac{\pi s}{2} \int_0^\infty dx\, x^{-s-1} \ln\left[\sqrt{2\pi x}\, e^{-x} I_\nu(x)\right], \quad \text{for } -1 < \text{Re } s < 0. \tag{3.8}$$

[3] In the mathematical literature, this object taken at even integer s is sometimes referred to as the Rayleigh function [33].

Yet, we shall have to work out (3.8) in order to obtain an equivalent representation more amenable to numerical calculation. The first step is to rescale $x \to \nu x$. Afterwards, we will perform a subtraction procedure on the resulting expression. The aim of that is the reduction of $\zeta_\nu^{\mathrm{I},\mathcal{D}}(s)$ to some elementary functions of s plus an integral, whose integrand should be relatively easy to express in terms of uniform asymptotic expansions[4] (u.a.e.'s).

The piece which we will subtract and add to the integrand of (3.8) is

$$x^{-s-1} \ln\left[\frac{\sqrt{x}}{(1+x^2)^{1/4}} e^{\nu(\eta(x)-x)}\right] = x^{-s-1} \quad \left[\sigma_1^{\mathrm{I},\mathcal{D}} \ln\frac{(1+x^2)^{1/4}}{\sqrt{x}} + \sigma_2^{\mathrm{I},\mathcal{D}}\nu(\eta(x)-x)\right], \qquad \sigma_1^{\mathrm{I},\mathcal{D}} = -1, \qquad \sigma_2^{\mathrm{I},\mathcal{D}} = +1. \tag{3.9}$$

where $\eta(x)$ is the function appearing in the known u.a.e. of the Bessel function (see e.g.[46]):

$$\eta(x) = \sqrt{1+x^2} + \ln\frac{x}{1+\sqrt{1+x^2}}, \tag{3.10}$$

and $\sigma_1^{\mathrm{I},\mathcal{D}}$, $\sigma_2^{\mathrm{I},\mathcal{D}}$ have been introduced for future convenience.

The added term is then separately integrated using the intermediate steps

$$\begin{aligned}
\int_0^\infty dx\, x^{-s-1} \ln\frac{(1+x^2)^{1/4}}{\sqrt{x}} &= \frac{\pi}{4s\sin\frac{\pi s}{2}}, \\
\int_0^\infty dx\, x^{-s-1}\,(\eta(x)-x) &= \int_0^\infty dx\, x^{-s-1} \ln\frac{x}{1+\sqrt{1+x^2}} + \int_0^\infty dx\ x^{-s-1}(\sqrt{1+x^2}-x) \\
&= \frac{1}{2s}B\left(\frac{s+1}{2},-\frac{s}{2}\right) + 2^{s-1}\left[B\left(\frac{s+1}{2},-s\right)+B\left(\frac{s+3}{2},-s\right)\right],
\end{aligned} \tag{3.11}$$

which lead to

$$\begin{aligned}
\zeta_\nu^{\mathrm{I},\mathcal{D}}(s) &= \frac{1}{4}\sigma_1^{\mathrm{I},\mathcal{D}}\nu^{-s} \\
&+\nu^{-s}\frac{s}{\pi}\sin\frac{\pi s}{2}\left[\sigma_2^{\mathrm{I},\mathcal{D}}\left\{\frac{1}{2s}B\left(\frac{s+1}{2},-\frac{s}{2}\right)+2^{s-1}B\left(\frac{s+1}{2},-s\right)\right.\right. \\
&\qquad\left. +2^{s-1}B\left(\frac{s+3}{2},-s\right)\right\}\nu \\
&\qquad\left. +\int_0^\infty dx\ x^{-s-1}\ln\left[\mathcal{L}^{\mathrm{I},\mathcal{D}}(\nu,x)\right]\right],
\end{aligned} \tag{3.12}$$

where, as usual $B(x,y) \equiv \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, and

$$\mathcal{L}^{\mathrm{I},\mathcal{D}}(\nu,x) \equiv \sqrt{2\pi\nu}(1+x^2)^{1/4}e^{-\nu\eta(x)}I_\nu(\nu x). \tag{3.13}$$

The advantage of this new representation is that the u.a.e. of the ln argument has reduced to

$$\mathcal{L}^{\mathrm{I},\mathcal{D}}(\nu,x) \sim 1 + \sum_{k=1}^\infty \frac{u_k(t(x))}{\nu^k}, \quad t(x) = \frac{1}{\sqrt{1+x^2}}, \tag{3.14}$$

where the u_k's are known polynomials listed in books like [46]. Expression (3.12) is also handy by the way in which the pole at $s=-1$ is exhibited. The singularity at this point is caused by:

i) the B functions with one argument equal to $\frac{s+1}{2}$

ii) the large-x behaviour of the integrand; by (3.14) one has

$$\ln\left[\mathcal{L}^{\mathrm{I},\mathcal{D}}(\nu,x)\right] = \frac{1}{8\nu}t(x) + O(t^2(x)) = \frac{1}{8\nu x} + O\left(\frac{1}{x^2}\right)$$

[4] also called Debye expansions

which asymptotically yields a logarithmic divergence. In fact, after integrating it has the same look as i), since

$$\int_0^\infty dx\, x^{-s-1}\, t(x)^m = \frac{1}{2} B\left(\frac{s+m}{2}, -\frac{s}{2}\right), \tag{3.15}$$

in this case with $m = 1$.

The calculation of the integral in (3.12) can be mentally divided into two stages. First: *necessarily* one has to delete and separately add the part responsible for the divergence in ii). Second: to make things numerically easier, and eventually extract some infinities which will appear later in the 'complete' zeta function, it is convenient to keep subtracting and adding more terms of the expansion of $\ln[\mathcal{L}^{\mathrm{I},\mathcal{D}}(\nu, x)]$

Thus, taking several terms in the series, we write the ln function in (3.12) as

$$\ln\left[\mathcal{L}^{\mathrm{I},\mathcal{D}}(\nu, x)\right] \sim \ln\left[1 + \sum_{k\geq 1} \frac{u_k(t(x))}{\nu^k}\right] = \sum_{n\geq 1} \frac{\mathcal{U}_n^{\mathrm{I},\mathcal{D}}(t(x))}{\nu^n}, \tag{3.16}$$

where the $\mathcal{U}_n^{\mathrm{I},\mathcal{D}}$'s are given by

$$\begin{aligned}
\mathcal{U}_1^{\mathrm{I},\mathcal{D}}(t) &= \frac{t}{8} - \frac{5\,t^3}{24}, \\
\mathcal{U}_2^{\mathrm{I},\mathcal{D}}(t) &= \frac{t^2}{16} - \frac{3\,t^4}{8} + \frac{5\,t^6}{16}, \\
\mathcal{U}_3^{\mathrm{I},\mathcal{D}}(t) &= \frac{25\,t^3}{384} - \frac{531\,t^5}{640} + \frac{221\,t^7}{128} - \frac{1105\,t^9}{1152}, \\
\mathcal{U}_4^{\mathrm{I},\mathcal{D}}(t) &= \frac{13\,t^4}{128} - \frac{71\,t^6}{32} + \frac{531\,t^8}{64} - \frac{339\,t^{10}}{32} + \frac{565\,t^{12}}{128}, \\
&\ \vdots
\end{aligned} \tag{3.17}$$

From them, we form the quantities

$$\mathcal{J}_n^{\mathrm{I},\mathcal{D}}(s) \equiv \int_0^\infty dx\, x^{-s-1}\, \mathcal{U}_n^{\mathrm{I},\mathcal{D}}(t(x)). \tag{3.18}$$

The expressions for these $\mathcal{J}_n^{\mathrm{I},\mathcal{D}}(s)$'s are easily obtained from the $\mathcal{U}_n^{\mathrm{I},\mathcal{D}}(t)$'s in (3.17) by application of (3.15), as a result of which it is enough to make the replacement

$$\begin{aligned}
\mathcal{U}_n^{\mathrm{I},\mathcal{D}}(t) &\to \mathcal{J}_n^{\mathrm{I},\mathcal{D}}(s) \\
t^m &\to \frac{1}{2} B\left(\frac{s+m}{2}, -\frac{s}{2}\right)
\end{aligned} \tag{3.19}$$

With this, we may write

$$\begin{aligned}
\int_0^\infty dx\, x^{-s-1}\, \ln\left[\mathcal{L}^{\mathrm{I},\mathcal{D}}(\nu, x)\right] &= \mathcal{S}_N^{\mathrm{I},\mathcal{D}}(s, \nu) + \sum_{n=1}^{N} \frac{\mathcal{J}_n^{\mathrm{I},\mathcal{D}}(s)}{\nu^n}, \\
\mathcal{S}_N^{\mathrm{I},\mathcal{D}}(s, \nu) \equiv \int_0^\infty dx\, x^{-s-1} &\left\{\ln\left[\mathcal{L}^{\mathrm{I},\mathcal{D}}(\nu, x)\right] - \sum_{n=1}^{N} \frac{\mathcal{U}_n^{\mathrm{I},\mathcal{D}}(t(x))}{\nu^n}\right\},
\end{aligned} \tag{3.20}$$

the key point being that $S_N^{\mathrm{I},\mathcal{D}}(s, \nu)$ is a *finite* integral at $s = -1$. Of all the $\mathcal{J}_n^{\mathrm{I},\mathcal{D}}(s)$'s, $\mathcal{J}_1^{\mathrm{I},\mathcal{D}}(s)$ is special as it contains the only contribution to the $s = -1$ pole coming from the integral in (3.12) (i.e. the outcome of the 'first stage'; the rest is produced by the 'second stage'). For this reason we change the notation to

$$\mathcal{J}_1^{\mathrm{I},\mathcal{D}}(s) = \frac{\rho^{\mathrm{I},\mathcal{D}}}{2} B\left(\frac{s+1}{2}, -\frac{s}{2}\right) + \bar{\mathcal{J}}_1^{\mathrm{I},\mathcal{D}}(s), \qquad \rho^{\mathrm{I},\mathcal{D}} = \frac{1}{8}, \qquad \bar{\mathcal{J}}_1^{\mathrm{I},\mathcal{D}}(s) = -\frac{5}{24} B\left(\frac{s+3}{2}, -\frac{s}{2}\right). \tag{3.21}$$

Here the symbol $\rho^{\mathrm{I},\mathcal{D}}$ has been introduced to make easier the expressions of the forthcoming cases. Then, we have

$$
\begin{aligned}
\zeta_\nu^{\mathrm{I},\mathcal{D}}(s) \;=\;& \frac{1}{4}\sigma_1^{\mathrm{I},\mathcal{D}}\nu^{-s} \\
&+\nu^{-s}\frac{s}{\pi}\sin\frac{\pi s}{2}\left[\sigma_2^{\mathrm{I},\mathcal{D}}\left\{\frac{1}{2s}B\left(\frac{s+1}{2},-\frac{s}{2}\right)+2^{s-1}B\left(\frac{s+1}{2},-s\right)\right.\right. \\
&\qquad\left. +2^{s-1}B\left(\frac{s+3}{2},-s\right)\right\}\nu \\
&\qquad+\mathcal{S}_N^{\mathrm{I},\mathcal{D}}(s,\nu) \\
&\qquad\left.+\frac{1}{2}\rho^{\mathrm{I},\mathcal{D}}B\left(\frac{s+1}{2},-\frac{s}{2}\right)\frac{1}{\nu}+\bar{\mathcal{J}}_1^{\mathrm{I},\mathcal{D}}(s)\frac{1}{\nu}+\sum_{n=2}^{N}\mathcal{J}_n^{\mathrm{I},\mathcal{D}}(s)\frac{1}{\nu^n}\right].
\end{aligned}
\tag{3.22}
$$

Laurent-expanding near $s=-1$ we find $\zeta_\nu(s)=\dfrac{1-4\nu^2}{8\pi}\dfrac{1}{s+1}+O((s+1)^0)$, which gives the right value of the residue at this point [34, 30, 31]. Similarly, by Taylor expanding close to $s=0$ we get $\zeta_\nu(s)=-\dfrac{1}{2}\left(\nu+\dfrac{1}{2}\right)+O(s)$, i.e., although we started from a representation valid for $-1<\mathrm{Re}\, s<0$, the correct value of ζ_ν at $s=0$ is also recovered.

3.1.2 'Complete' zeta function

Next, we go on to the D-dimensional problem. This is done by inserting both (3.6) and (3.22) into the first formula of (3.5). Afterwards, we may trivially interchange the l-summation and the k-summation, since the second is finite. Once we have done so, we are left with sums of the type

$$
\sum_{l=l_{\min}}^{\infty}\nu(D,l)^{-z}=\zeta_H\left(z,\frac{D}{2}-1+l_{\min}\right),
\tag{3.23}
$$

(recall (2.1)) ζ_H denoting the Hurwitz zeta function. Thus, we arrive at

$$
\begin{aligned}
&\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{D}}(s)=\frac{2a^s}{(D-2)!}\sum_{k=0}^{k_{\max}(D)}(-1)^k\mathcal{A}_k(D) \\
&\times\left[\frac{1}{4}\sigma_1^{\mathrm{I},\mathcal{D}}\zeta_H\left(-D+2+2k+s,\frac{D}{2}-1+l_{\min}\right)\right. \\
&\quad+\frac{s}{\pi}\sin\frac{\pi s}{2}\left\{\sigma_2^{\mathrm{I},\mathcal{D}}\left(\frac{1}{2s}B\left(\frac{s+1}{2},-\frac{s}{2}\right)+2^{s-1}B\left(\frac{s+1}{2},-s\right)+2^{s-1}B\left(\frac{s+3}{2},-s\right)\right)\right. \\
&\qquad\quad\times\zeta_H\left(-D+1+2k+s,\frac{D}{2}-1+l_{\min}\right) \\
&\qquad\quad+\sum_{l=l_{\min}}^{\infty}\mathcal{S}_N^{\mathrm{I},\mathcal{D}}(s,\nu(D,l))\nu(D,l)^{D-2-2k-s} \\
&\qquad\quad+\frac{1}{2}\rho^{\mathrm{I},\mathcal{D}}B\left(\frac{s+1}{2},-\frac{s}{2}\right)\zeta_H\left(-D+3+2k+s,\frac{D}{2}-1+l_{\min}\right) \\
&\qquad\quad+\bar{\mathcal{J}}_1^{\mathrm{I},\mathcal{D}}(s)\zeta_H\left(-D+3+2k+s,\frac{D}{2}-1+l_{\min}\right) \\
&\qquad\quad\left.\left.+\sum_{n=2}^{N}\mathcal{J}_n^{\mathrm{I},\mathcal{D}}(s)\zeta_H\left(-D+2+2k+s+n,\frac{D}{2}-1+l_{\min}\right)\right\}\right].
\end{aligned}
\tag{3.24}
$$

Examining again the origins of singularity at $s=-1$, we find

a) The ones already present for ζ_ν, visible now as $\sim B\left(\frac{s+1}{2},\ldots\right)$.

b) New pole contributions when the first argument of any of the present Hurwitz zeta functions equals one, including the terms with n and k such that $n=D-2k$ in the last sum.

On the other hand, at $s=-1$, the series $\sum_{l=l_{\min}}^{\infty} \mathcal{S}_N^{\mathrm{I},\mathcal{D}}(s,\nu(D,l))\,\nu(D,l)^{D-2-2k-s}$ appears to have a rather slow numerical convergence but, since its net contribution is actually little (and the larger N, the smaller it gets) in practice it suffices to compute until an accuracy of few digits is achieved.

3.1.3 Three-dimensional space

As a result of (3.7), for $D=3$ one has $k_{\max}(3)=0$ and the k-series in (3.24) reduces to the $k=0$-term, with $\mathcal{A}_0=1$. Further, we consider the description of the electromagnetic TE modes and therefore set $l_{\min}=1$ (for an ordinary scalar field $l_{\min}=0$). We split $\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{D}}(s)$ into two pieces: $\zeta_{\mathcal{M}1}^{\mathrm{I},\mathcal{D}}(s)$, containing the part directly evaluable at $s=-1$, and $\zeta_{\mathcal{M}2}^{\mathrm{I},\mathcal{D}}(s)$ which includes the 'mixed' terms with both singular and finite contributions. Then

$$\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{D}}(s)=\zeta_{\mathcal{M}1}^{\mathrm{I},\mathcal{D}}(s)+\zeta_{\mathcal{M}2}^{\mathrm{I},\mathcal{D}}(s), \tag{3.25}$$

where

$$\begin{aligned}\zeta_{\mathcal{M}1}^{\mathrm{I},\mathcal{D}}(s)=2a^s\quad&\left[\frac{1}{4}\sigma_1^{\mathrm{I},\mathcal{D}}\zeta_H\left(s-1,\frac{3}{2}\right)\right.\\&+\frac{s}{\pi}\sin\frac{\pi s}{2}\left\{\sigma_2^{\mathrm{I},\mathcal{D}}2^{s-1}B\left(\frac{s+3}{2},-s\right)\zeta_H\left(s-2,\frac{3}{2}\right)\right.\\&\qquad+\sum_{l\geq 1}\mathcal{S}_N^{\mathrm{I},\mathcal{D}}\left(s,l+\frac{1}{2}\right)\left(l+\frac{1}{2}\right)^{1-s}\\&\qquad\left.\left.+\bar{\mathcal{J}}_1^{\mathrm{I},\mathcal{D}}(s)\zeta_H\left(s,\frac{3}{2}\right)+\sum_{\substack{n=2\\n\neq 3}}^{N}\mathcal{J}_n^{\mathrm{I},\mathcal{D}}(s)\zeta_H\left(n+s-1,\frac{3}{2}\right)\right\}\right]\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\zeta_{\mathcal{M}2}^{\mathrm{I},\mathcal{D}}(s)=2a^s\sin\frac{\pi s}{2}\quad&\left[\sigma_2^{\mathrm{I},\mathcal{D}}\left\{\frac{1}{2s}B\left(\frac{s+1}{2},-\frac{s}{2}\right)+2^{s-1}B\left(\frac{s+1}{2},-s\right)\right\}\zeta_H\left(s-2,\frac{3}{2}\right)\right.\\&+\frac{1}{2}\rho^{\mathrm{I},\mathcal{D}}B\left(\frac{s+1}{2},-\frac{s}{2}\right)\zeta_H\left(s,\frac{3}{2}\right)\\&\left.+\ \mathcal{J}_3^{\mathrm{I},\mathcal{D}}(s)\zeta_H\left(s+2,\frac{3}{2}\right)\right]\end{aligned} \tag{3.27}$$

Next, we will compute $\zeta_{\mathcal{M}1}^{\mathrm{I},\mathcal{D}}(s)$ directly at $s=-1$ and Laurent-expand $\zeta_{\mathcal{M}2}^{\mathrm{I},\mathcal{D}}(s)$ near $s=-1$. With the specific values of $\sigma_1^{\mathrm{I},\mathcal{D}}$, $\sigma_2^{\mathrm{I},\mathcal{D}}$, $\rho^{\mathrm{I},\mathcal{D}}$ from (3.9), (3.21) and taking $N=4$ subtractions, we get

$$\zeta_{\mathcal{M}1}^{\mathrm{I},\mathcal{D}}(-1)=\frac{1}{a\pi}\left[\frac{719}{5760}+\frac{1053\,\pi}{8192}+\frac{35\,\pi^3}{65536}+2\sum_{l\geq 1}\mathcal{S}_4^{\mathrm{I},\mathcal{D}}\left(-1,l+\frac{1}{2}\right)\left(l+\frac{1}{2}\right)^2\right], \tag{3.28}$$

where, after a numerical calculation (with $N=4$), we have found

$$\sum_{l=1}^{\infty}\mathcal{S}_4^{\mathrm{I},\mathcal{D}}\left(-1,l+\frac{1}{2}\right)\left(l+\frac{1}{2}\right)^2\simeq 0.00024 \tag{3.29}$$

The part containing the singularities gives rise to the following Laurent expansion

$$\begin{aligned}\zeta_{\mathcal{M}2}^{\mathrm{I},\mathcal{D}}(s)=\frac{1}{a\pi}\quad&\left[\frac{2}{315}\left(\frac{1}{s+1}+\ln a\right)\right.\\&+\frac{18457}{60480}-\frac{229}{20160}\gamma-\frac{11}{672}\ln 2-\zeta_H'\left(-3,\frac{3}{2}\right)+\frac{1}{4}\zeta_H'\left(-1,\frac{3}{2}\right)\\&\left.+O(s+1)\right].\end{aligned} \tag{3.30}$$

Here the prime denotes derivative with respect to the first argument. To complete the desired numerical evaluation, we still need the values of $\zeta_H'\left(-3,\frac{3}{2}\right)$ and $\zeta_H'\left(-1,\frac{3}{2}\right)$, which are found from the relation $\zeta_H\left(s,\frac{3}{2}\right)=$

$-2^s+(2^s-1)\zeta_R(s)$ —where ζ_R means the Riemann zeta function— and from the knowledge of $\zeta_R'(-3) \simeq 0.005378$ and $\zeta_R'(-1) \simeq -0.165421$.

Equipped with all this, we are able to obtain

$$\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{D}}(s) = \frac{1}{a}\left[\frac{2}{315\pi}\left(\frac{1}{s+1} + \ln a\right) + 0.27069 + O(s+1)\right]. \tag{3.31}$$

The residue of the pole at $s = -1$ is $\mathrm{Res}\left[\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{D}}(s), s=-1\right] = \frac{2}{315\pi a}$, Bearing in mind that a 1/2-factor appears when going from the zeta function for the Maxwell eigenmodes to that for the Laplacian spectrum, we realize that this agrees with the heat-kernel expansion of that operator in the Dirichlet case (see e.g. [16]). The existence of this pole indicates that the Casimir energy under the present conditions is still infinite after zeta funtion regularization, and its divergence cannot be removed until the application of the PP prescription, which in some sense amounts to renormalizing. (The issue of infinities for the bag model was considered, from the cutoff viewpoint, in [47]. A full discussion about the essence of divergences in Casimir energy problems was supplied in [48].)

Repeating the calculation for $l_{\min} = 0$ one obtains

$$\zeta_{\mathcal{M}}^{\mathrm{scal},\mathrm{I},\mathcal{D}}(s) = \frac{1}{a}\left[\frac{2}{315\pi}\left(\frac{1}{s+1} + \ln a\right) + 0.00889 + O(s+1)\right] \tag{3.32}$$

which corresponds to the internal modes of a *true* scalar field. In fact, this value may also be found by adding to (3.31) the 'partial wave' contribution of the $l = 0$ mode alone, which is

$$\frac{1}{a}\zeta_{1/2}^{\mathrm{I},\mathcal{D}}(-1) = \frac{\pi}{a}\zeta_R(-1) = -\frac{\pi}{12a} \simeq -0.26180\frac{1}{a} \tag{3.33}$$

(Note here that, since $J_{1/2}(x) \propto x^{-1/2}\sin x$, $\zeta_{1/2}^{\mathrm{I},\mathcal{D}}(s) = \pi^{-s}\zeta_R(s)$). As one can check, the figures match. Moreover, the importance of the lower-lying region of the spectrum is manifest, since the $l = 0$ part is almost as large as the rest, but with opposite sign. An approximate calculation of (3.32) based on the heat-kernel expansion of the Laplacian is given in app. B.

3.1.4 Two-dimensional space

One can now think of the circular wire problem in the plane. Given that we are still dealing with a Dirichlet field, this part will not be needed for the $D = 2$ e.m. Casimir effect, and is done just for completeness. Now $d(2,l) = 2$ for $l > 0$ and $d(2,0) = 1$. With the methods in the preceding subsubsections, i.e. carefully taking (3.24) for $D = 2$ and $l_{\min} = 1$, we find:

$$\zeta_{\mathcal{M}}^{\mathrm{scal},\mathrm{I},\mathcal{D}}(s) = \frac{1}{a}\left[-\frac{16+\pi}{128\pi}\left(\frac{1}{s+1} + \ln a\right) + 0.02436 + O(s+1)\right]. \tag{3.34}$$

Observe that $\nu(2, l = 0) = 0$ stops us from making the rescaling $x \to \nu x$ and applying u.a.e.'s. A way of dealing with $l = 0$ parts in $D = 2$ will be shown later.

3.2 Internal Robin (TM) modes

Here we outline the changes when considering the eigenmodes obeying the Robin b.c. (2.4). The analogue of (3.8) for these conditions is (see [31])

$$\zeta_\nu^{\mathrm{I},\mathcal{R}}(s) = \frac{s}{\pi}\sin\frac{\pi s}{2}\int_0^\infty dx\; x^{-s-1}\ln\left[\sqrt{\frac{2\pi}{x}}e^{-x}\left(xI_\nu'(x) + \alpha\, I_\nu(x)\right)\right], \quad \text{for } -1 < \mathrm{Re}\; s < 0 \tag{3.35}$$

We shall take $\alpha(D) = D/2 - 1$ in accordance with (2.4) and thus $\alpha(3) = 1/2$. The calculation will be performed by a subtraction procedure similar to that for the TE modes but, now, the piece which we remove and add to the integrand is (instead of (3.9))

$$x^{-s-1} \ln\left[\frac{(1+x^2)^{1/4}}{\sqrt{x}} e^{\nu(\eta(x)-x)}\right] = x^{-s-1} \quad \left[\sigma_1^{\mathrm{I},\mathcal{R}} \ln \frac{(1+x^2)^{1/4}}{\sqrt{x}} + \sigma_2^{\mathrm{I},\mathcal{R}} \nu(\eta(x)-x)\right], \qquad \sigma_1^{\mathrm{I},\mathcal{R}} = +1, \qquad \sigma_2^{\mathrm{I},\mathcal{R}} = +1. \tag{3.36}$$

The ensuing expression is similar to (3.12) but for these changes:
a) The signs $\sigma_1^{\mathrm{I},\mathcal{D}}, \sigma_2^{\mathrm{I},\mathcal{D}}$ are replaced with $\sigma_1^{\mathrm{I},\mathcal{R}}, \sigma_2^{\mathrm{I},\mathcal{R}}$.
b) Instead of (3.13), for the present case one has

$$\mathcal{L}^{\mathrm{I},\mathcal{R}}(\nu, x) = \sqrt{2\pi\nu} \frac{1}{(1+x^2)^{1/4}} e^{-\nu\eta(x)} x I'_\nu(\nu x) + \alpha \frac{1}{\nu\sqrt{1+x^2}} \sqrt{2\pi\nu}(1+x^2)^{1/4} e^{-\nu\eta(x)} I_\nu(\nu x)$$

Then $\ln\left[\mathcal{L}^{\mathrm{I},\mathcal{R}}(\nu, x)\right]$ is expanded by taking advantage of the u.a.e.'s of both $I'_\nu(\nu x)$ and $I_\nu(\nu x)$ (see [46] again)

$$\ln\left[\mathcal{L}^{\mathrm{I},\mathcal{R}}(\nu, x)\right] \sim \ln\left[1 + \sum_{k\geq 1} \frac{v_k(t(x))}{\nu^k} + \alpha \frac{t(x)}{\nu}\left(1 + \sum_{k\geq 1} \frac{u_k(t(x))}{\nu^k}\right)\right] = \sum_{n\geq 1} \frac{\mathcal{U}_n^{\mathrm{I},\mathcal{R}}(t(x))}{\nu^n}$$

where

$$\begin{aligned}
\mathcal{U}_1^{\mathrm{I},\mathcal{R}}(t) &= \left(-\frac{3}{8} + \alpha\right) t + \frac{7\,t^3}{24}, \\
\mathcal{U}_2^{\mathrm{I},\mathcal{R}}(t) &= \left(-\frac{3}{16} + \frac{\alpha}{2} - \frac{\alpha^2}{2}\right) t^2 + \left(\frac{5}{8} - \frac{\alpha}{2}\right) t^4 - \frac{7\,t^6}{16}, \\
\mathcal{U}_3^{\mathrm{I},\mathcal{R}}(t) &= \left(-\frac{21}{128} + \frac{3\,\alpha}{8} - \frac{\alpha^2}{2} + \frac{\alpha^3}{3}\right) t^3 + \left(\frac{869}{640} - \frac{5\,\alpha}{4} + \frac{\alpha^2}{2}\right) t^5 + \left(-\frac{315}{128} + \frac{7\,\alpha}{8}\right) t^7 \\
&\quad + \frac{1463\,t^9}{1152}, \\
\mathcal{U}_4^{\mathrm{I},\mathcal{R}}(t) &= \left(-\frac{27}{128} + \frac{3\,\alpha}{8} - \frac{\alpha^2}{2} + \frac{\alpha^3}{2} - \frac{\alpha^4}{4}\right) t^4 + \left(\frac{109}{32} - \frac{23\,\alpha}{8} + \frac{3\,\alpha^2}{2} - \frac{\alpha^3}{2}\right) t^6 \\
&\quad + \left(-\frac{733}{64} + \frac{41\,\alpha}{8} - \alpha^2\right) t^8 + \left(\frac{441}{32} - \frac{21\,\alpha}{8}\right) t^{10} - \frac{707\,t^{12}}{128}, \\
&\vdots
\end{aligned} \tag{3.37}$$

These formulas parallel eqs. (3.16) and (3.17). Looking at the linear t-term in the first of (3.37) we realize that the analogue of $\rho^{\mathrm{I},\mathcal{D}}$ in the previous case (eq. (3.21)) is

$$\rho^{\mathrm{I},\mathcal{R}} = \alpha - \frac{3}{8}. \tag{3.38}$$

The expression for $\zeta_\nu^{\mathrm{I},\mathcal{R}}(s)$ is like (3.22), replacing all the quantities with $\mathrm{I},\mathcal{D}$-superscripts by the corresponding ones obtained with $\mathrm{I},\mathcal{R}$-superscripts. When doing so, we must bear in mind that in the case under consideration

$$\mathcal{S}_N^{\mathrm{I},\mathcal{R}}(s, \nu) = \int_0^\infty dx\, x^{-s-1} \left\{\ln\left[\mathcal{L}^{\mathrm{I},\mathcal{R}}(\nu, x)\right] - \sum_{n=1}^{N} \frac{\mathcal{U}_n^{\mathrm{I},\mathcal{R}}(t(x))}{\nu^n}\right\} \tag{3.39}$$

(which is by construction a finite integral) and

$$\bar{\mathcal{J}}_1^{\mathrm{I},\mathcal{R}}(s) = -\frac{5}{24} B\left(\frac{s+3}{2}, -\frac{s}{2}\right), \qquad \mathcal{J}_n^{\mathrm{I},\mathcal{R}}(s) = \int_0^\infty dx\, x^{-s-1}\, \mathcal{U}_n^{\mathrm{I},\mathcal{R}}(t(x)), \ n \geq 2 \tag{3.40}$$

For studying the complete spherical problem, we must deal with the $\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{R}}(s)$ in the second line of (3.5), and apply the same formula (3.6) for the $d(D, l)$'s, obtaining a similar series of Hurwitz functions, integrals, etc. Later, we

specialize it to $D = 3$, (therefore $\alpha = 1/2$) and $l_{\min} = 1$. Before going on, we separate $\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{R}}(s) = \zeta_{\mathcal{M}1}^{\mathrm{I},\mathcal{R}}(s) + \zeta_{\mathcal{M}2}^{\mathrm{I},\mathcal{R}}(s)$ following the same finiteness criterion as in the decomposition (3.25). Then, the resulting $\zeta_{\mathcal{M}1}^{\mathrm{I},\mathcal{R}}(s)$, $\zeta_{\mathcal{M}2}^{\mathrm{I},\mathcal{R}}(s)$ are obtained from the $\zeta_{\mathcal{M}1}^{\mathrm{I},\mathcal{D}}(s)$, $\zeta_{\mathcal{M}2}^{\mathrm{I},\mathcal{D}}(s)$ in formulas (3.26), (3.27) by just replacing all the objects having $\mathrm{I},\mathcal{D}$-superscript by their counterparts with $\mathrm{I},\mathcal{R}$-superscript. We also need the numerical calculation

$$\sum_{l=1}^{\infty} \mathcal{S}_4^{\mathrm{I},\mathcal{R}}\left(-1, l+\frac{1}{2}\right)\left(l+\frac{1}{2}\right)^2 \simeq 0.00012 \tag{3.41}$$

(analogous to (3.29), and done for $N = 4$ too). After adding $\zeta_{\mathcal{M}1}^{\mathrm{I},\mathcal{R}}(-1)$ and the Laurent-expansion of $\zeta_{\mathcal{M}2}^{\mathrm{I},\mathcal{R}}(s)$ near $s = -1$, we find

$$\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{R}}(s) = \frac{1}{a}\left[\frac{2}{45\pi}\left(\frac{1}{s+1} + \ln a\right) - 0.10285 + O(s+1)\right]. \tag{3.42}$$

For a scalar field, one can either repeat the calculation with $l_{\min} = 0$ or separately add the contribution from this mode. The second option is easy because, using $J_{1/2}(x) \propto x^{-1/2}\sin x$, our Robin condition for $\nu(3,0) = 1/2$ just reads $x^{1/2}\cos x = 0$. Its nonvanishing solutions are $\pi\left(n+\frac{1}{2}\right), n = 0, 1, 2, \ldots$ and therefore $\zeta_{\nu=1/2}^{\mathrm{I},\mathcal{R}}(s) = \pi^{-s}\zeta_H\left(s, \frac{1}{2}\right)$. At $s = -1$ one has

$$\frac{1}{a}\pi\zeta_H\left(-1, \frac{1}{2}\right) = \frac{\pi}{24a} = 0.13090\frac{1}{a}. \tag{3.43}$$

So, after adding this part,

$$\zeta_{\mathcal{M}}^{\mathrm{scal},\mathrm{I},\mathcal{R}}(s) = \frac{1}{a}\left[\frac{2}{315\pi}\left(\frac{1}{s+1} + \ln a\right) + 0.02805 + O(s+1)\right]. \tag{3.44}$$

The situation $D = 2$, $\alpha = 0$ may be similarly studied, arriving at the complete zeta function for the Neumann modes with $l_{\min} = 1$:

$$\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{N}}(s) = \zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{R}}(s)\Big|_{\alpha=0} = \frac{1}{a}\left[\frac{48-5\pi}{128\pi}\left(\frac{1}{s+1} + \ln a\right) + 0.17883 + O(s+1)\right]. \tag{3.45}$$

4 External modes

As already commented, the spectrum of modes —ω's— is determined by the effect of the problem's conditions on the radial part of the wave solutions (in QFT language we would say 'the *field* solutions'). This applies both to the solutions in the interior (region I) and to those outside the spherical surface (region II), which shall be now considered. There are at least two possible approaches, both of them leading to the same result:

1. First, we sketch the simplest one. If we demand that the external solutions behave like outgoing radial waves $\sim e^{i\omega r}$ for $r \to \infty$, (though the discussion might be repeated with asymptotically ingoing partial waves as well) their radial parts will be just $\propto r^{1-D/2} H_{\nu(D,l)}^{(1)}(\omega r)$, with $H_\nu^{(1)}$ denoting the first Hankel function (ingoing waves would just be the complex conjugate, thus involving $H^{(1)\,*} = H^{(2)}$). Following this cue, we repeat the contour integration procedure of ref. [30], which gave the representations (3.8), (3.35), but putting now $H_\nu^{(1)}$ instead of J_ν at the outset. Since the calculations have the same nature, we do not go throught them here; perhaps the only point worthy of remark is that wherever in ref.[31] we took advantage of properties like

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right), \ |z| \gg 1, \ |\arg z| < \pi,$$
$$J_\nu(e^{i\pi/2}z) = e^{i\nu\pi/2} I_\nu(z), \ -\pi < \arg z \leq \frac{\pi}{2},$$

now we have to make use of

$$
\begin{aligned}
H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i\left(z-\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right)},\ |z| \gg 1,\ -\pi < \arg z < 2\pi, \\
H_\nu^{(1)}(e^{i\pi/2}z) = -i\frac{2}{\pi}e^{-i\nu\pi/2}K_\nu(z),\ -\pi < \arg z \leq \frac{\pi}{2},
\end{aligned}
$$

(and their equivalents for conjugates and derivatives). Doing so, one obtains formulas (4.2) and (4.11) below.

2. Another reasoning, physically more transparent, is to imagine a larger sphere of radius R, enclosing the one of radius a, on whose surface we impose conditions as well (in the sipirit of ref.[41]). Once their partial-wave zeta function has been constructed, the $R \to \infty$ limit is obtained. Not only is the result R-independent, but it coincides with the outcome of the method 1 as well. The full process is explained in app. A.

Once the desired 'partial-wave' zeta functions $\zeta_\nu^{\mathrm{II},\mathcal{D}}(s)$, $\zeta_\nu^{\mathrm{II},\mathcal{R}}(s)$ have been obtained by either method, taking the Bessel index $\nu(D,l)$ and the degeneracy $d(D,l)$ of each l in D dimensions, we construct the 'complete' spherical zeta functions

$$
\begin{aligned}
\zeta_{\mathcal{M}}^{\mathrm{II},\mathcal{D}}(s) &= a^s \sum_{l=l_{\min}}^{\infty} d(D,l)\, \zeta_{\nu(D,l)}^{\mathrm{II},\mathcal{D}}(s), \\
\zeta_{\mathcal{M}}^{\mathrm{II},\mathcal{R}}(s) &= a^s \sum_{l=l_{\min}}^{\infty} d(D,l)\, \zeta_{\nu(D,l)}^{\mathrm{II},\mathcal{R}}(s).
\end{aligned} \tag{4.1}
$$

When considering the e.m. field, we will eventually set $l_{\min} = 1$ (for a scalar one, $l_{\min} = 0$).

4.1 Dirichlet (TE) external modes

The adequate representation of the 'partial-wave' zeta function analogous to (3.8) for region II is

$$
\zeta_\nu^{\mathrm{II},\mathcal{D}}(s) = \frac{s}{\pi}\sin\frac{\pi s}{2}\int_0^\infty dx\, x^{-s-1}\, \ln\left[\sqrt{\frac{2x}{\pi}}\, e^x K_\nu(x)\right], \quad \text{for } -1 < \mathrm{Re}\ s < 0. \tag{4.2}
$$

After rescaling $x \to \nu x$, a subtraction procedure similar to that for the internal modes will take place, but now the piece which we remove and add to the integrand is

$$
x^{-s-1}\ln\left[\frac{\sqrt{x}}{(1+x^2)^{1/4}}e^{-\nu(\eta(x)-x)}\right] \tag{4.3}
$$

As compared to (3.9), this amounts to a sign flip in ν. Therefore

$$
\sigma_1^{\mathrm{II},\mathcal{D}} = -1, \qquad \sigma_2^{\mathrm{II},\mathcal{D}} = -1.
$$

The ensuing expression is similar to (3.12) but with the above σ_1, σ_2 instead of those, and the function $\mathcal{L}^{\mathrm{I},\mathcal{D}}(\nu,x)$ appearing there replaced now with

$$
\mathcal{L}^{\mathrm{II},\mathcal{D}}(\nu,x) = \sqrt{\frac{2\nu}{\pi}}\,(1+x^2)^{1/4}e^{\nu\eta(x)}K_\nu(\nu x).
$$

$\ln\left[\mathcal{L}^{\mathrm{II},\mathcal{D}}(\nu,x)\right]$ is expanded by taking adavantage of the u.a.e. of $K_\nu(\nu x)$ (see [46] again)

$$
\ln\left[\mathcal{L}^{\mathrm{II},\mathcal{D}}(\nu,x)\right] \sim \ln\left[1+\sum_{k\geq 1}(-1)^k\frac{u_k(t(x))}{\nu^k}\right] = \sum_{n\geq 1}\frac{\mathcal{U}_n^{\mathrm{II},\mathcal{D}}(t(x))}{\nu^n},
$$

where, by obvious parity reasoning with respect to the $\mathrm{I},\mathcal{D}$ case (compare the above expression with (3.16))

$$
\mathcal{U}_n^{\mathrm{II},\mathcal{D}}(t) = (-1)^n\mathcal{U}_n^{\mathrm{I},\mathcal{D}}(t). \tag{4.4}
$$

As a result of (4.4)

$$\rho^{\mathrm{II},\mathcal{D}} = -\rho^{\mathrm{I},\mathcal{D}} = -\frac{1}{8}. \tag{4.5}$$

Hence, by virtue of (3.18) and of (4.4),

$$\bar{\mathcal{J}}_1^{\mathrm{II},\mathcal{D}}(s) = -\bar{\mathcal{J}}_1^{\mathrm{I},\mathcal{D}}(s), \qquad \mathcal{J}_n^{\mathrm{II},\mathcal{D}}(s) = (-1)^n \mathcal{J}_n^{\mathrm{I},\mathcal{D}}(s),\ n \geq 2. \tag{4.6}$$

With these elements, we can already construct the analogue of (3.22), say $\zeta_\nu^{\mathrm{II},\mathcal{D}}(s)$, for the external Dirichlet modes by 'superscript substitution'. The resulting expression contains

$$\mathcal{S}_N^{\mathrm{II},\mathcal{D}}(s,\nu) = \int_0^\infty dx\, x^{-s-1} \left\{ \ln\left[\mathcal{L}^{\mathrm{II},\mathcal{D}}(\nu,x)\right] - \sum_{n=1}^{N} \frac{\mathcal{U}_n^{\mathrm{II},\mathcal{D}}(t(x))}{\nu^n} \right\} \tag{4.7}$$

which is a finite integral. When studying the complete spherical problem, we must handle the $\zeta_{\mathcal{M}}^{\mathrm{II},\mathcal{D}}(s)$ in the first line of (4.1), and apply again formula (3.6) for the $d(D,l)$'s. Then, the emerging complete zeta function is like (3.24), but with all $\mathrm{I},\mathcal{D}$-superscripts turned into $\mathrm{II},\mathcal{D}$-superscripts. After setting $D=3$, $l_{\min}=1$, we Laurent-expand around $s=-1$, finding that everything is of the same sort as in the internal case, except for the new quantity $\sum_{l=1}^{\infty} \mathcal{S}_4^{\mathrm{II},\mathcal{D}}\left(-1, l+\frac{1}{2}\right)\left(l+\frac{1}{2}\right)^2 \simeq -0.00054$ (here calculated for $N=4$ too). So, after gathering everything together one arrives at

$$\zeta_{\mathcal{M}}^{\mathrm{II},\mathcal{D}}(s) = \frac{1}{a}\left[-\frac{2}{315\pi}\left(\frac{1}{s+1} + \ln a\right) - 0.00326 + O(s+1)\right]. \tag{4.8}$$

Redoing the calculation for $l_{\min}=0$, corresponding to a scalar field, we obtain $\zeta_{\mathcal{M}}^{\mathrm{scal},\mathrm{II},\mathcal{D}}(s) = \zeta_{\mathcal{M}}^{\mathrm{II},\mathcal{D}}(s)$, i.e. the $l=0$ mode is not contributing in this case.

When $D=2$, $l_{\min}=1$, we get

$$\zeta_{\mathcal{M}}^{\mathrm{II},\mathcal{D}}(s) = \frac{1}{a}\left[\frac{16-\pi}{128\pi}\left(\frac{1}{s+1} + \ln a\right) + 0.00501 + O(s+1)\right]. \tag{4.9}$$

Now, the net contribution from the $l=0$ Dirichlet mode in $D=2$ will be found by joining the inner and outer partial wave zeta functions (3.8), (4.2). This sum has the effect of cancelling the $s=-1$ divergences as a result of which we can numerically integrate, obtaining:

$$\frac{1}{a}\left[\zeta_{\nu=0}^{\mathrm{I},\mathcal{D}}(-1) + \zeta_{\nu=0}^{\mathrm{II},\mathcal{D}}(-1)\right] = \frac{1}{a\pi}\int_0^\infty dx\, \ln[-2xI_0(x)K_0(x)] = -0.02802\frac{1}{a}. \tag{4.10}$$

Actually, by a calculation based on a slightly different representation of $\zeta_\nu^{\mathrm{I},\mathcal{D}}(s)$ valid for $\nu=0$ [32], we know that the part of (4.10) coming from the internal modes is

$$\frac{1}{a}\zeta_{\nu=0}^{\mathrm{I},\mathcal{D}}(s) = \frac{1}{a}\left[\frac{1}{8\pi}\frac{1}{s+1} - 0.01451 + O(s+1)\right].$$

4.2 Robin (TM) external modes

Applying any of the above referred procedures we find the 'partial-wave' zeta function representation

$$\zeta_\nu^{\mathrm{II},\mathcal{R}}(s) = \frac{s}{\pi}\sin\frac{\pi s}{2}\int_0^\infty dx\, x^{-s-1}\ \ln\left[\sqrt{\frac{2}{\pi x}}\, e^x(-xK_\nu'(x) - \alpha K_\nu(x))\right], \quad \text{for } -1 < \mathrm{Re}\ s < 0. \tag{4.11}$$

The subtracted part will be

$$x^{-s-1}\ln\left[\frac{(1+x^2)^{1/4}}{\sqrt{x}} e^{-\nu(\eta(x)-x)}\right] \tag{4.12}$$

Thus, paralleling (3.36),

$$\sigma_1^{\mathrm{II},\mathcal{R}} = +1, \qquad \sigma_2^{\mathrm{II},\mathcal{R}} = -1.$$

With respect to the situation described as $\mathrm{I},\mathcal{R}$, this amounts to a sign change in ν and the corresponding parity reasonings apply everywhere (as when going from $\mathrm{I},\mathcal{D}$ to $\mathrm{II},\mathcal{D}$). In the present case we have to deal with

$$\begin{aligned}\mathcal{L}^{\mathrm{II},\mathcal{R}}(\nu,x) &= -\sqrt{\frac{2\nu}{\pi}}\frac{1}{(1+x^2)^{1/4}}e^{\nu\eta(x)}xK_\nu'(\nu x) - \alpha\frac{1}{\nu\sqrt{1+x^2}}\sqrt{\frac{2\nu}{\pi}}(1+x^2)^{1/4}e^{\nu\eta(x)}K_\nu(\nu x), \\ \mathcal{S}_N^{\mathrm{II},\mathcal{R}}(s,\nu) &= \int_0^\infty dx\, x^{-s-1}\left\{\ln\left[\mathcal{L}^{\mathrm{II},\mathcal{R}}(\nu,x)\right] - \sum_{n=1}^N \frac{\mathcal{U}_n^{\mathrm{II},\mathcal{R}}(t(x))}{\nu^n}\right\},\end{aligned} \tag{4.13}$$

where $\mathcal{U}_n^{\mathrm{II},\mathcal{R}}(t) = (-1)^n\mathcal{U}_n^{\mathrm{I},\mathcal{R}}(t)$ and therefore $\rho^{\mathrm{II},\mathcal{R}} = -\rho^{\mathrm{I},\mathcal{R}}$, $\bar{\mathcal{J}}_1^{\mathrm{II},\mathcal{R}}(s) = -\bar{\mathcal{J}}_1^{\mathrm{I},\mathcal{R}}(s)$, $\mathcal{J}_n^{\mathrm{II},\mathcal{R}}(s) = (-1)^n\mathcal{J}_n^{\mathrm{I},\mathcal{R}}(s)$, $n \geq 2$.

Then, we construct the complete zeta function, which we calculate for $D = 3, \alpha = 1/2$. Taking into account that (for $N = 4$) $\sum_{l=1}^{\infty}\mathcal{S}_4^{\mathrm{II},\mathcal{R}}\left(-1, l+\frac{1}{2}\right)\left(l+\frac{1}{2}\right)^2 \simeq -0.00041$, we are able to write the Laurent expansion near $s = -1$:

$$\zeta_{\mathcal{M}}^{\mathrm{II},\mathcal{R}}(s) = \frac{1}{a}\left[-\frac{2}{45\pi}\left(\frac{1}{s+1} + \ln a\right) - 0.07223 + O(s+1)\right]. \tag{4.14}$$

As for a possible $l = 0$ contribution when considering a scalar field we see, like in the external Dirichlet case, that this mode changes nothing, i.e. $\zeta_{\mathcal{M}}^{\mathrm{scal},\mathrm{II},\mathcal{R}}(s) = \zeta_{\mathcal{M}}^{\mathrm{II},\mathcal{R}}(s)$.

When $D = 2$, $\alpha = 0$,

$$\zeta_{\mathcal{M}}^{\mathrm{II},\mathcal{N}}(s) = \zeta_{\mathcal{M}}^{\mathrm{II},\mathcal{R}}(s)\Big|_{\alpha=0} = \frac{1}{a}\left[-\frac{48+5\pi}{128\pi}\left(\frac{1}{s+1} + \ln a\right) - 0.03804 + O(s+1)\right]. \tag{4.15}$$

As in the $D = 2$ Dirichlet case, the total contribution from the $l = 0$ mode alone follows from adding the inner and outer partial wave zeta functions which are now (3.35) and (4.11). Again, the $s = -1$ divergences cancel and we can find, by numerical integration:

$$\frac{1}{a}\left[\zeta_{\nu=0}^{\mathrm{I},\mathcal{R}}(-1) + \zeta_{\nu=0}^{\mathrm{II},\mathcal{R}}(-1)\right]_{\alpha=0} = \frac{1}{a\pi}\int_0^\infty dx\, \ln[-2xI_0'(x)K_0'(x)] = -0.50704\frac{1}{a} \tag{4.16}$$

5 Discussion

5.1 $D = 3$

One can consider the contribution to the e.m. Casimir energy coming from the interior of the sphere only, i.e. the zero-point energy of a photon bag banning the existence of outer modes. Taking into account (3.2) and the sum of (3.31) plus (3.42) one finds

$$E_{\mathrm{C}}^{\mathrm{e.m.I}}(\mu) = \frac{1}{a}\left[\frac{8}{315\pi}\ln(a\mu) + 0.08392\right]. \tag{5.1}$$

The logarithmic term, depending on μ, has to be viewed as a remainder of the renormalization process implicit in the prescription adopted [6]. The second article of refs. [20] gives, for the Casimir energy due to the vector gauge bosons in the interior of such a bag

$$\frac{1}{a}\left[\frac{8}{315\pi}\ln\frac{\delta}{8} + 0.08984\right], \tag{5.2}$$

found by the ‘energy method’ of [13]. δ is a cutoff arising from the non-coincidence in time of field points, linkable to the nonzero ‘skin depth’ of a real —and not purely mathematical— surface. Incidentally, Riemann

zeta functions were also employed in part of the calculations in [20], although the initial regularization approach was essentially different from ours. Therefore, altough they are within a 6.7 %, there is no deep reason why the finite parts in (5.1) and (5.2) should be equal, as figures may vary by just changing the values of the different cutoffs. In a (fermionless) QCD context μ can arguably be related to the momentum scale parameter $\Lambda_{\rm QCD}$ (see e.g. [49] and [6]). Results for that model follow by just bringing in an obvious (N_c^2-1)-factor from the SU(N_c) degrees of freedom; thus, for $N_c = 3$ the cutoff-independent part of the energy becomes $\sim \frac{0.7}{a}$. (When fermions are assumed to be massless, neglecting them is not too bad an approximation, since results in the second work of refs.[20] showed their contribution as being one order of magnitude smaller).

Back to the e.m. case, the same confining set-up —without external modes— in cubic cavities (see [50, 3]) yields a Casimir energy that amounts to $0.0916/L$, where L is the edge length. The resemblance among this number and the finite parts of (5.1) , (5.2) happens to be striking, although in view of the different details in the schemes leading to their derivation one ought to be cautious before taking this point any further. Joining (3.32) and $\zeta_{\mathcal{M}}^{\rm scal,II,\mathcal{D}}(s)$, which is equal to (4.8), we shall find the net Casimir energy for a scalar field filling the whole space and satisfying Dirichlet b.c. on the spherical surface. *Whithout* having to apply (3.2), one gets a finite and scale-independent result, which reads

$$E_{\rm C}^{\rm scal.\ \mathcal{D}} = \frac{1}{a}\, 0.00282 \qquad (5.3)$$

This finiteness is due to the cancellation of both poles at $s = -1$ when adding up internal and external contributions, and may be put down to the self-erasing of curvature-dependent infinities with same size but opposite sign on each side of the surface. (5.3) coincides with the energy value which would yield the force (3.24) of ref. [10] for the same physical situation, derived from a Green function approach.

Finally we consider an e.m. field in the whole space with the sphere acting as a neutral and perfectly conducting boundary, which corresponds to the sum of the four results (3.31), (3.42), (4.8), (4.14). On addition, we realize that the poles cancel within the $\mathcal{D}$-pair and within the $\mathcal{R}$-pair separately, rendering the PP prescription in (3.2) unnecessary as we are left with a finite and scale-independent value, namely

$$E_{\rm C}^{\rm e.m.} = \frac{1}{2a}\, 0.09235 \qquad (5.4)$$

that coincides with the celebrated figure of [41, 14, 13].

Let's take another glance at all the results for $E_{\rm C}$ *prior* to the application of PP. The outcome is summarized in table 1. This way one easily sees that the PP prescription is redundant when any internal-external pair is

$D=3$	$\mathcal{D}$irichlet	$\mathcal{R}$obin ($\alpha(3)=1/2$)
l=0 (only region I contributes)	$\frac{1}{2a}\left[-\frac{\pi}{12}\right]$	$\frac{1}{2a}\,\frac{\pi}{24}$
$\{l\geq 1\}$ region I	$\frac{1}{2a}\left[\frac{2}{315}\left(\frac{1}{s+1}+\ln(a\mu)\right)+0.27069\right]$	$\frac{1}{2a}\left[\frac{2}{45}\left(\frac{1}{s+1}+\ln(a\mu)\right)-0.10285\right]$
$\{l\geq 1\}$ region II	$\frac{1}{2a}\left[-\frac{2}{315}\left(\frac{1}{s+1}+\ln(a\mu)\right)-0.00326\right]$	$\frac{1}{2a}\left[-\frac{2}{45}\left(\frac{1}{s+1}+\ln(a\mu)\right)-0.07223\right]$

Table 1: Zero-point energy decomposition in terms of scalar fields satisfying Dirichlet and Robin b.c. on a spherical surface of radius a in $D = 3$.

added up, which may be envisaged as the above commented surface divergence cancellation. An analysis of

heat-kernel coefficients allows one to realize that these singularities are odd —and therefore of opposite sign on each side— when the space dimension is an odd number. In zeta regularization this is no longer so for curved surfaces in even D (see [6, 52] — notice also that the infinities in [10] confirm this observation). This fact has to be faced, in particular, in $D = 2$.

5.2 $D = 2$

Although we have argued that the e.m. problem in $D = 2$ reduces to a Neumann field, for completeness the analogous figures associated to a Dirichlet field were also calculated, and all the values obtained have been listed in table 2. Adding up all the Neumann parts, which are in the second column, one finds

$D = 2$	$\mathcal{D}$irichlet	$\mathcal{N}$eumann ($\alpha(2) = 0$)
l=0 (regions I + II)	$\frac{1}{2a}\left[-0.02802\right]$	$\frac{1}{2a}\left[-0.50704\right]$
$\{l \geq 1\}$ region I	$\frac{1}{2a}\left[-\frac{16+\pi}{128\pi}\left(\frac{1}{s+1}+\ln(a\mu)\right) +0.02436\right]$	$\frac{1}{2a}\left[\frac{48-5\pi}{128\pi}\left(\frac{1}{s+1}+\ln(a\mu)\right) +0.17883\right]$
$\{l \geq 1\}$ region II	$\frac{1}{2a}\left[\frac{16-\pi}{128\pi}\left(\frac{1}{s+1}+\ln(a\mu)\right) +0.00501\right]$	$\frac{1}{2a}\left[-\frac{48+5\pi}{128\pi}\left(\frac{1}{s+1}+\ln(a\mu)\right) -0.03804\right]$

Table 2: Zero-point energy decomposition in scalar fields under Dirichlet and Neumann conditions on a circular line of radius a in $D = 2$

$$E_{\mathrm{C}}^{\mathrm{e.m.}}(\mu) = \frac{1}{a}\left[-\frac{5}{128}\ln(a\mu) - 0.18312\right]. \tag{5.5}$$

This minus sign makes us think of the attractive force obtained for a cylinder in [54] (although that refers to three-dimensional space, the symmetry is the same). Not even the sum of inner and outer parts cancels all the singularities. This is a consequence of using zeta regularization with a curved boundary in even space dimension [6, 52].

Doing the same for a scalar Dirichlet field we have

$$E_{\mathrm{C}}^{\mathrm{scal.}\ \mathcal{D}}(\mu) = \frac{1}{a}\left[-\frac{1}{128}\ln(a\mu) + 0.00068\right]. \tag{5.6}$$

The coefficient of the logarithmic part agrees with the first ref. of [51] for the same set-up. Since the regularization method in that paper was frequency cutoff, this coefficient is all that one should expect to coincide. About possible ambiguities coming from the use of different regularization schemes, see e.g. the comments in [53].

A comparison with the results in ref. [37] is now in order. The contributions from the $l = 0$ mode in the Dirichlet (scalar) and Neumann (e.m.) cases agree with formulas (A13) and (3.5) —respectively— in that work. However, after performing the sum for infinite values of l, a divergent part $-\frac{5}{128a}\frac{1}{s+1}$ —formula (5.5) prior to prescription— comes into existence through the pole of $\zeta_H(z,1) = \zeta_R(z)$ when z equals one (i.e. what we have called, below eq.(3.24), a b-type singularity). Now, this specific piece has survived the infinity cancellations which take place when adding all the internal and external parts. Observing the parity of all our $\mathcal{J}_n(s)$'s when going from internal to external parts, it is not difficult to realize that this happens for even $D \geq 2$.

These poles were also detected in sect. III of ref. [10] after using dimensional regularization and employing Riemann zeta functions in the last stage of the calculation. Within our zeta-regularization context, this divergence

would invalidate the alleged reliability of the finite estimates in ref. [37] for the $l \neq 0$ mode contribution (formulas (A16) and (3.9) of that paper, for scalar and e.m. cases, respectively). The existence of a singularity was there acknowledged, but it was argued that it could be a 'spurious' one. In our own framework, that speculation seems to be more questionable. By way of connecting results, we show in app. C how to reobtain those estimates by performing what might be called deliberately 'naïve' zeta regularization.

5.3 Ending comments

Canonical field quantization leads to operators —e.g. the Hamiltonian— with ill-defined vacuum expectation values. Such troubles, due to quantum fluctuations, are often suppressed by decree removing them when no observable effects are expected. The picture is different in the presence of external sources or boundaries that, by breaking symmetries, render fluctuations observable. This explains the longevity of Casimir's general concept of vacuum energy, according to which the physical vacuum of quantum fields must be determined including their constraints.

Full vacuum energies may contain nonobvious infinite pieces originated in boundary surface tensions, but our attention here has focused on the parts responsible for Casimir forces, i.e. those containing dependence on the relevant space parameter, which may be finite in spite of a global energy singularity. That is the way in which 'finiteness' has to be qualified [48].

Viewed as an evaluation method for the Casimir effect, zeta function regularization had been relatively successful up to now but it was leaving some vague aftertaste of scepticism insofar as it was applied to comparatively simple problems: parallel plates, hypercubes, torii, hyperspheres as framework spaces (not as boundaries), i.e. situations where eigenvalues are at worst polynomials in the quantum numbers. Circular or spherical boundaries are beyond this realm. The unified approach of the present work has quickly enabled us to recover

- three remarkable Casimir energy results involving a sphere:

    1. e.m. field inside: second row of table 1, which gives the coefficient for the logarithmic term in formula (5.1) as in [20]. The closeness of the remaining pieces may be in principle fortuitous since they come from different regularizations (but is numerically convenient for comparing scales). That is why we should regard our finite part as a new zeta-regularization result.
    2. scalar Dirichlet field inside and outside: first column in table 1, producing (5.3) like in ref. [10].
    3. e.m. field inside and outside: second plus third rows in table 1, that yield (5.4) coinciding with refs. [41, 14, 13],

- and one concerning a circle:

    1. scalar Dirichlet field inside and outside: first column in table 2, giving (5.6) whose logarithm coefficent is the same as in [51]. The rest is a new result of zeta function regularization.

Furthermore, combining figures other findings emerge, e.g. the sum of all the contributions for the $D = 3$ Robin scalar field —second column in table 1— gives $E_{\mathrm{C}}^{\mathrm{scal.}\ \mathcal{R}} = -0.02209\frac{1}{a}$, of a larger order of magnitude than and opposite sign to its Dirichlet counterpart (5.4). Moreover, the zeta-function regularized version of the e.m. Casimir energy inside and outside a circle (5.5) is, to our knowledge, another unreleased result.

We hope that the new answers provided —together with the recovery of figures originally obtained after considerable effort— by this single zeta function strike can give the reader reasons to think that the versatility and scope of this technique may be somewhat wider.

A Appendix: external modes

A.1 External Dirichlet modes

Here we give a detailed proof of expression (4.2). Our starting point is the defining series of the zeta function for a scalar field existing between two spheres. The inner one has radius a and is assumed to be physical. The outer one has radius R and it is introduced for the sake of convenience; eventually we shall take the $R \to \infty$ limit. Taking into account that $\nu(D=3,l) = l + \frac{1}{2}$, initially we take solutions with radial part $Ah_l^{(1)}(\lambda r) + Bh_l^{(2)}(\lambda r)$, where the h_l's are spherical Hankel functions and A, B are coefficients whose relative value —with respect to each other— will be determined by imposing our Dirichlet conditions on both surfaces, i.e. at $r = a$ and $r = R$. Doing so, we are left with an homogeneous linear system for (A, B) and, by requiring its compatibility, the equation

$$f_{a,R}(\lambda) \equiv h_l^{(1)}(\lambda a) h_l^{(2)}(\lambda R) - h_l^{(1)}(\lambda R) h_l^{(2)}(\lambda a) = 0$$

follows. Therefore we study the 'partial-wave' zeta function $\zeta_\nu(z) = \sum_p \frac{1}{\lambda_{\nu,p}^z}$, where $\lambda_{\nu,p}$ is the p-th zero of the function $f_{a,R}(\lambda)$.

We tread here along the same lane that was open in [31, 30], which runs through

$$\zeta_\nu(z) = \frac{z}{2\pi i} \int d\lambda \;\; \lambda^{-z-1} \ln\left(f_{a,R}(\lambda)\right). \tag{A.1}$$

In expression (A.1) the integration contour winds counterclockwise round the zeros of $f_{a,R}$, which are real. As it stands, this gives a representation for ζ_ν valid if Re $z > 1$. We slightly modify our representation to

$$\zeta_\nu(z) = \frac{z}{2\pi i} \int d\lambda \;\; \lambda^{-z-1} \ln\left(\frac{i}{\lambda} \frac{\nu a^n R^n}{R^{2\nu} - a^{2\nu}} f_{a,R}(\lambda)\right), \tag{A.2}$$

which ensures that the argument of the logarithm goes to 1 when λ approaches 0; this will be seen to be useful in the sequel.

Now we deform the contour in such a way that it is made up of three parts: a straight line from $+i\infty$ to $i\epsilon$, an arch of radius ϵ connecting $i\epsilon$ to $-i\epsilon$, and a straight line from $-i\epsilon$ to $-i\infty$. The courteous reader may care to see in [46] that

$$f_{a,R}(\lambda) = e^{i\lambda a - i\lambda R} S_n\left(-i\lambda a\right) S_n\left(i\lambda R\right) - e^{-i\lambda a + i\lambda R} S_n\left(-i\lambda R\right) S_n\left(i\lambda a\right), \tag{A.3}$$

where $S_n\left(z\right) = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} (2z)^{-k}$.

With this in mind, let us try to find an analytic continuation to the contribution from the upper straight line. We immediately see that the dominant contribution in the argument of the logarithm comes from the $e^{i\lambda a - i\lambda R}$ term. So, we separate this factor in the logarithm and apply the property that the logarithm of a product equals the sum of the logarithms of its factors. This leads to

$$\begin{aligned} \zeta_{\nu,+}(z) = -\frac{z}{2\pi i^{z+1}} \Bigg\{ (R-a) \frac{\epsilon^{1-z}}{z-1} \\ + \int_\epsilon^\infty \frac{d\rho}{\rho^{z+1}} \ln\left[\frac{1}{\rho} \frac{\nu a^n R^n}{R^{2\nu} - a^{2\nu}} \left(S_n(\rho a) S_n(-\rho R) - e^{-2\rho(R-a)} S_n(\rho R) S_n(-\rho a)\right)\right] \Bigg\}. \end{aligned} \tag{A.4}$$

This expression explicitly defines an analytic function for Re $z > 0$. There is an equivalent representation for $\zeta_{\nu,-}$ (the contribution from the lower straight line). It reads

$$\zeta_{\nu,-}(z) = -\frac{z i^{z+1}}{2\pi} \left\{ (R-a) \frac{\epsilon^{1-z}}{z-1} \right.$$
$$\left. + \int_\epsilon^\infty \frac{d\rho}{\rho^{z+1}} \ln \left[\frac{1}{\rho} \frac{\nu a^n R^n}{R^{2\nu} - a^{2\nu}} \left(S_n(\rho a) S_n(-\rho R) - e^{-2\rho(R-a)} S_n(\rho R) S_n(-\rho a) \right) \right] \right\}. \tag{A.5}$$

The complete ζ_ν is given by the addition of these contributions plus the integration along the small arch of radius ϵ. In any case we have now an explicit continuation valid for Re $z > 0$. If we restrict to the domain $0 <$ Re $z < 1$, and take the limit $\epsilon \to 0$ we end up with

$$\zeta_\nu(z) = \frac{z}{\pi} \sin\left(\frac{\pi}{2} z\right) \int_0^\infty \frac{d\rho}{\rho^{z+1}} \ln \left[\frac{1}{\rho} \frac{\nu a^n R^n}{R^{2\nu} - a^{2\nu}} \left(S_n(\rho a) S_n(-\rho R) - e^{-2\rho(R-a)} S_n(\rho R) S_n(-\rho a) \right) \right]. \tag{A.6}$$

By now it may seem unpleasant that the large-R limit is ill-defined. When we go on with the continuation process towards domains in the complex set with negative real part it will be apparent that in those regions one may perform this limit, which will allow us to extract physical results.

In order to go on we have to add and subtract a function that apes the asymptotic behaviour of the integrand for large λ and which may be analytically integrated, this is the general procedure which was set forth in [30, 31]. Nevertheless, before we do this, we proceed to further simplify our expressions. First of all, we separate the integral into three parts:

$$\begin{aligned}
\zeta_\nu(z) \;=\; & \frac{z}{\pi} \int_L^\infty \frac{d\rho}{\rho^{z+1}} \ln \left[S_n(\rho a) S_n(-\rho R) - e^{-2\rho(R-a)} S_n(\rho R) S_n(-\rho a) \right] \\
& + \frac{z}{\pi} \int_L^\infty \frac{d\rho}{\rho^{z+1}} \ln \left[\frac{1}{\rho} \frac{\nu a^n R^n}{R^{2\nu} - a^{2\nu}} \right] \\
& + \frac{z}{\pi} \int_0^L \frac{d\rho}{\rho^{z+1}} \left(\ln \left[\frac{1}{\rho} \frac{\nu a^n R^n}{R^{2\nu} - a^{2\nu}} \right] + \ln \left[S_n(\rho a) S_n(-\rho R) - e^{-2\rho(R-a)} S_n(\rho R) S_n(-\rho a) \right] \right) \\
\equiv\; & \zeta_{\nu,1}(z) + \zeta_{\nu,2}(z) + \zeta_{\nu,3}(z),
\end{aligned} \tag{A.7}$$

where L is any positive number. The second integration is trivial and gives a meromorphic function with a unique pole at $z = 0$. The third integration directly defines an analytic function in Re $z < 1$. It is the first integration which calls for the special treatment that has been overviewed above. So, we write

$$\zeta_{\nu,1}(z) = \frac{z}{\pi} \sin\left(\frac{\pi}{2} z\right)$$
$$\times \int_L^\infty \frac{d\rho}{\rho^{z+1}} \left(\ln \left[\frac{S_n(\rho a) S_n(-\rho R) - e^{-2\rho(R-a)} S_n(\rho R) S_n(-\rho a)}{A_n(\rho a, \rho R)} \right] + \ln \left(A_n(\rho a, \rho R) \right) \right), \tag{A.8}$$

where $A_n(\rho a, \rho R)$ is a function which shares the asymptotic behaviour for large ρ with

$$S_n(\rho a) S_n(-\rho R) - e^{-2\rho(R-a)} S_n(\rho R) S_n(-\rho a)$$

in such a way that

$$\frac{z}{\pi} \sin\left(\frac{\pi}{2} z\right) \int_L^\infty \frac{d\rho}{\rho^{z+1}} \ln \left[\frac{S_n(\rho a) S_n(-\rho R) - e^{-2\rho(R-a)} S_n(\rho R) S_n(-\rho a)}{A_n(\rho a, \rho R)} \right] \tag{A.9}$$

defines an analytic function for $-z_o <$ Re z (where z_o is a positive number which can be made as large as we wish by dint of complicating A_n), and

$$\frac{z}{\pi} \sin\left(\frac{\pi}{2} z\right) \int_L^\infty \frac{d\rho}{\rho^{z+1}} \ln A_n(\rho a, \rho R) \tag{A.10}$$

is easily computed, giving an explicit meromorphic function. Once this is achieved we have our much coveted analytic continuation of ζ_ν for Re $z > -z_o$. In this setting, we may restrict the function ζ_ν to $-z_o <$ Re $z < 0$; in this domain we have

$$\zeta_\nu(z) = \zeta_{\nu,1}(z) + \frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^L \frac{d\rho}{\rho^{z+1}} \ln\left(S_n(\rho a)S_n(-\rho R) - e^{-2\rho(R-a)}S_n(\rho R)S_n(-\rho a)\right) \quad (A.11)$$

as $\zeta_{\nu,2}$ happens to cancel one contribution from $\zeta_{\nu,3}$. To sum up, we write $(-z_o < z < 0)$

$$\begin{aligned} \zeta_\nu(z) &= \frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_L^\infty \frac{d\rho}{\rho^{z+1}} \ln\left[\frac{S_n(\rho a)S_n(-\rho R) - e^{-2\rho(R-a)}S_n(\rho R)S_n(-\rho a)}{A_n(\rho a, \rho R)}\right] \\ &+\frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^L \frac{d\rho}{\rho^{z+1}} \ln\left(S_n(\rho a)S_n(-\rho R) - e^{-2\rho(R-a)}S_n(\rho R)S_n(-\rho a)\right) \\ &+\text{Analytic continuation of}\left[\frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_L^\infty \frac{d\rho}{\rho^{z+1}} \ln A_n(\rho a, \rho R)\right]. \end{aligned} \quad (A.12)$$

We have specified that we should calculate the analytic continuation in the last term, as the integral

$$\frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_L^\infty \frac{d\rho}{\rho^{z+1}} \ln A_n(\rho a, \rho R) \quad (A.13)$$

only exists for Re $z > -1$. Now, if $A_n(\rho a, \rho R)$ also satisfies that expression (A.13) exists in the domain $0 <$ Re $z < 1$ and its analytic continuation may be easily computed in the particular case that $L = 0$, then expression (A.12) may be simplified to

$$\begin{aligned} \zeta_\nu(z) &= \frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}} \ln\left[\frac{S_n(\rho a)S_n(-\rho R) - e^{-2\rho(R-a)}S_n(\rho R)S_n(-\rho a)}{A_n(\rho a, \rho R)}\right] \\ &+\text{Analytic continuation of}\left[\frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}} \ln A_n(\rho a, \rho R)\right]. \end{aligned} \quad (A.14)$$

The sagacious reader will immediately recognize that this expression admits a simple large-R limit, which may be written as

$$\begin{aligned} \zeta_\nu(z) &= \frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}} \ln\left[\frac{S_n(\rho a)}{A_n(\rho a)}\right] \\ &+\text{Analytic continuation of}\left[\frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}} \ln A_n(\rho a)\right], \end{aligned} \quad (A.15)$$

where $A_n(\rho a)$ is supposed to have the asymptotic behaviour of $S_n(\rho a)$ to some order. Note that this limit has been made possible when we have passed to regions with negative real part. To finish this digression, we cast expression (A.15) into a more standard form:

$$\begin{aligned} \zeta_\nu(z) &= \frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}} \ln\left[\frac{\sqrt{\frac{2\rho a}{\pi}}e^{\rho a}K_{n+\frac{1}{2}}(\rho a)}{A_n(\rho a)}\right] \\ &+\text{Analytic continuation of}\left[\frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}} \ln A_n(\rho a)\right], \end{aligned} \quad (A.16)$$

If we only need a representation valid in $-1 <$ Re $z < 0$ we need not include any A_n at all and the result is

$$\zeta_\nu(z) = \frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}} \ln\left(\sqrt{\frac{2\rho a}{\pi}}e^{\rho a}K_{n+\frac{1}{2}}(\rho a)\right) \quad (A.17)$$

which, when rescaled to $a = 1$, and for $n + 1/2 = \nu(D = 3, n)$, gives (4.2).

A.2 External Robin modes

We explain now how to arrive at the expression for Robin boundary condition in the external region.

In this derivation we will keep in mind what we did for the modes satisfying Dirichlet boundary conditions. So we shall also consider the existence of a large spherical shell of radius R which will be taken to ∞ once we have performed the analytic continuation to domains with Re $z < 0$.

Note that we will be assuming here that the conditions that the modes satisfy on the outer sphere are also of the same Robin type as those which are imposed on the physical one. This is equivalent to saying that the space is confined by a large sphere which is a perfect conductor. In any case this particular choice of boundary conditions on the outer sphere has no physical relevance in the sense that when one has performed the analytic continuation and $R \to \infty$, the results do not depend on it, for instance, we might as well decide to impose Dirichlet boundary conditions for both TE and TM modes on this unphysical sphere, and the final result (A.25) would not be changed at all.

Setting Robin conditions at $r = a$ and $r = R$ on the radial part of the spherical wave solution, one is led to consider the $\lambda_{\nu(3,l),p}$'s which are zeros of the function

$$f_{a,R}(\lambda) \equiv \sigma_l^{(1)}(\lambda a)\sigma_l^{(2)}(\lambda R) - \sigma_l^{(1)}(\lambda R)\sigma_l^{(2)}(\lambda a), \tag{A.18}$$

where $\sigma_l^{(i)}(z) = h_n^{(i)}(z) + z\frac{d}{dz}h_n^{(i)}(z)$ and $\nu(3,l) = l + \frac{1}{2}$. In more detail one has

$$\sigma_n^{(1)}(z) = i^{-n-2}e^{iz}\left(-S_n(-iz) + S_n^{'}(-iz)\right)$$

$$\sigma_n^{(2)}(z) = i^{n+2}e^{-iz}\left(-S_n(iz) + S_n^{'}(iz)\right).$$

As we did in the case of Dirichlet boundary conditions we first confine our efforts to the study of $\zeta_\nu(z) = \sum_p \frac{1}{\lambda_{\nu,p}^z}$. which is again given by (A.1), but now with the $f_{a,R}(\lambda)$ in eq. (A.18). The integration contour winds counterclockwise round the poles of $f_{a,R}$, which are real (this property is easily drawn from the fact that these poles are eigenvalues of a self-adjoint operator). Our complex integral is a valid representation for ζ_ν if Re $z > 1$. We slightly modify it by inserting a harmless factor in the argument of the logarithm, which is again contrived so as to make the log an infinitesimal quantity when λ approaches the origin of the complex plane

$$\zeta_\nu(z) = \frac{z}{2\pi i}\int d\lambda\ \lambda^{-z-1}\ln\left(\frac{i}{\lambda}\frac{\nu\lambda a^{n+1}R^{n+1}}{n(n+1)i(R^{2\nu} - a^{2\nu})}f_{a,R}(\lambda)\right), \tag{A.19}$$

this is valid if $n > 0$.

Now we deform the contour the same way as we did before: a straight line from $+i\infty$ to $i\epsilon$, an arch of radius ϵ connecting $i\epsilon$ to $-i\epsilon$, and a straight line from $-i\epsilon$ to $-i\infty$. Now the procedure closely follows what we did in the case of TE modes. We find three contributions: $\zeta_{\nu,+}$, $\zeta_{\nu,-}$ and the contribution from the arch. Both $\zeta_{\nu,+}$, $\zeta_{\nu,-}$ are given by

$$\zeta_{\nu,+}(z) = \frac{zi^{-z-1}}{2\pi}\frac{\epsilon^{1-z}}{1-z}(R-a) - \frac{zi^{-z-1}}{2\pi}\int_\epsilon^\infty \frac{d\rho}{\rho^{z+1}}\ln\left[\frac{\nu\rho}{n(n+1)}\frac{a^{n+1}R^{n+1}}{R^{2\nu} - a^{2\nu}}e^{-\rho(R-a)}f_{a,R}(i\rho)\right] \tag{A.20}$$

$$\zeta_{\nu,-}(z) = \frac{zi^{z+1}}{2\pi}\frac{\epsilon^{1-z}}{1-z}(R-a) - \frac{zi^{z+1}}{2\pi}\int_\epsilon^\infty \frac{d\rho}{\rho^{z+1}}\ln\left[\frac{\nu\rho}{n(n+1)}\frac{a^{n+1}R^{n+1}}{R^{2\nu} - a^{2\nu}}e^{-\rho(R-a)}f_{a,R}(i\rho)\right]. \tag{A.21}$$

It is immediate that we have realized an analytic continuation for Re $z > 0$.

If we restrict the domain to $0 < \mathrm{Re}\ z < 1$, and take the limit $\epsilon \to 0$ we end up with (note that in this limit the contribution from the arch is vanishingly small)

$$\zeta_\nu(z) = \frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}}\ln\left[\frac{-\nu\rho a^{n+1}R^{n+1}}{n(n+1)(R^{2\nu}-a^{2\nu})}f_{a,R}(-i\rho)e^{-\rho(R-a)}\right]. \tag{A.22}$$

For the sake of clarity we give here

$$\begin{aligned} -e^{-(R-a)}f_{a,R}(-i\rho) &= \left(S_n(\rho a)-S_n^{'}(\rho a)\right)\left(S_n(-\rho R)-S_n^{'}(-\rho R)\right) \\ &\quad -e^{-2(R-a)}\left[\left(S_n(\rho R)-S_n^{'}(\rho R)\right)\left(S_n(-\rho a)-S_n^{'}(-\rho a)\right)\right]. \end{aligned} \tag{A.23}$$

Now the reader should have no difficulty to apply the same procedure that we have explained in the case of external TE modes. In particular, we will have that for any z_o there is a proper function $A_n(\rho a, \rho R)$ such that the analytic continuation of ζ_ν for $-z_o < \mathrm{Re}\ z < 0$ is given by

$$\begin{aligned} \zeta_\nu(z) &= \text{Analytic continuation of } \left[\frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}}\ln A_n(\rho a,\rho R)\right] \\ &\quad +\frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}}\ln\left[\frac{-e^{-(R-a)}f_{a,R}(-i\rho)}{A_n(\rho a,\rho R)}\right]. \end{aligned} \tag{A.24}$$

The choice of function A_n is not uniquely determined. By simple inspection one sees that it would be enough a function such that for large ρ, $A_n(\rho a, \rho R)$ has the same (algebraic) asymptotic behaviour (to some convenient order) as the product

$$\left(S_n(\rho a)-S_n^{'}(\rho a)\right)\left(S_n(-\rho R)-S_n^{'}(-\rho R)\right).$$

A proper choice might be of the form $A_n(\rho a,\rho R) = A_n^{(1)}(\rho a)A_n^{(2)}(\rho R)$, where $A_n^{(1)}(\rho a)$ is an algebraic function that asymptotically behaves like

$$S_n(\rho a)-S_n^{'}(\rho a)$$

and $A_n^{(2)}(\rho R)$ does the same job for

$$S_n(-\rho R)-S_n^{'}(-\rho R).$$

This would suffice for our purposes. As $\lim_{R\to\infty} A_n^{(2)}(\rho R) = 1$, which is easily concluded from its asymptotic behaviour, we see that in the domain which we are considering it is possible the $R \to \infty$ limit. This limit leads to

$$\begin{aligned} \zeta_\nu(z) &= \text{Analytic continuation of } \left[\frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}}\ln A_n^{(1)}(\rho a)\right] \\ &\quad +\frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}}\ln\left[\frac{S_n(\rho a)-S_n^{'}(\rho a)}{A_n^{(1)}(\rho a)}\right]. \end{aligned} \tag{A.25}$$

If only a continuation valid in $-1 < \mathrm{Re}\ z < 0$ is needed, then we simply have

$$\zeta_\nu(z) = \frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}}\ln\left[S_n(\rho a)-S_n^{'}(\rho a)\right] \tag{A.26}$$

and, for $a = 1$,

$$\zeta_\nu(z) = \frac{z}{\pi}\sin\left(\frac{\pi}{2}z\right)\int_0^\infty \frac{d\rho}{\rho^{z+1}}\ln\left[\sqrt{\frac{2}{\pi z}}e^{z}\left(-zK_\nu^{'}(z)-\frac{1}{2}K_\nu(z)\right)\right], \tag{A.27}$$

which gives (4.11) after noting that $\alpha(D = 3) = 1/2$.

B Appendix: heat-kernel series approximation to the complete spectral zeta function

For the Laplacian $\Box$ (or any such operator satisfying some suitable requirements) the spectral zeta function $\zeta_\Box(z) = \text{Tr}\ \Box^{-z}$ is related to the heat kernel $Y_\Box(t) = \text{Tr}\ e^{-t\Box}$ through the well-known Mellin transform

$$\zeta_\Box(z) = \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1}\, Y_\Box(t). \tag{B.1}$$

In turn, the heat kernel has small-t asymptotic expansion

$$Y_\Box(t) \sim \frac{1}{(4\pi t)^{D/2}} \sum_{k\geq 0} b_{k/2} t^{k/2}. \tag{B.2}$$

We split the integration domain into the intervals $[0,\tau]$ and $[\tau,\infty)$, where τ is such that (B.2) holds for $t \leq \tau$. Thus, one can use this series in the first part and integrate, finding

$$\zeta_\Box(z) = \frac{1}{\Gamma(z)} \left[\frac{1}{(4\pi t)^{D/2}} \sum_{k\geq 0} \frac{b_{k/2}\, \tau^{z+k/2-D/2}}{z+k/2-D/2} + \int_\tau^\infty dt\, t^{z-1}\, Y_\Box(t) \right]. \tag{B.3}$$

In our case, where the boundary is a sphere, the values of the $b_{k/2}$'s can be read off from [18] (up to $k = 20$ if necessary!). Since $\Box$ gives squares of e.m. modes, symbolically $\Box = \mathcal{M}^2$ and $\zeta_{\mathcal{M}}(s) = \zeta_\Box(s/2)$, which shall be studied around $s = -1$. Thus, separating the singular part at this point we can write

$$\zeta_{\mathcal{M}}(s) = \frac{r}{s+1} + p(\tau) + c(\tau) + O(s+1), \tag{B.4}$$

where

$$\begin{aligned} r &= \frac{2b_{(D+1)/2}}{(4\pi)^{D/2}\Gamma\left(-\frac{1}{2}\right)}, \\ p(\tau) &= \frac{1}{(4\pi)^{D/2}\Gamma\left(-\frac{1}{2}\right)} \left[b_{(D+1)/2}(\ln\tau - \psi(-1/2)) + 2 \sum_{\substack{k\geq 0 \\ k\neq D+1}} \frac{b_{k/2}\tau^{(k-D-1)/2}}{k-D-1} \right], \\ c(\tau) &= \frac{1}{(4\pi)^{D/2}\Gamma\left(-\frac{1}{2}\right)} \int_\tau^\infty dt\, t^{-3/2}\, Y_\Box(t). \end{aligned} \tag{B.5}$$

From heat-kernel properties, one also knows that for large t

$$Y_\Box(t) \sim e^{-\lambda_0 t}, \tag{B.6}$$

where λ_0 stands for the smallest eigenvalue of $\Box$. We wish to use this behaviour in order to approximate the value of $c(\tau)$.

As an example, we take $D = 3$ and a scalar Dirichlet field inside the sphere. From the b_2 listed in e.g. [18], we find

$$r = \frac{2}{315\pi} \tag{B.7}$$

as we already knew ((3.32)). λ_0 will be the square of the smallest nonvanishing zero of $J_{1/2}(x)$ which is $x = \pi$; therefore $\lambda_0 = \pi^2$. Now, a delicate question arises about how to choose τ large enough so that the replacement (B.6) be sensible while maintaining the validity of the small-t expansion (B.2) for $t \in [0,\tau]$. The answer will necessarily be a compromise between both requirements. Looking at the integral $c(\tau)$, which we hope to keep small, one considers the region near $t = \tau$ and realizes that it would be desirable to have $\lambda_0\tau \gg 1$. Thus

largest k	approx. $p(1/2)$
5	0.009230
10	0.009331
15	0.009333

Table 3: Approximations to $p(1/2)$, depending on the largest k included in the sum.

one arrives at $\frac{1}{\lambda_0} \ll \tau < 1$. Since $\lambda_0 = \pi^2$, we deem $\tau = 1/2$ as a fairly adequate choice. Then we seek an approximation to $c(1/2)$, which is shown by means of table 3. On the other hand, the approximate value of the integral is, by (B.6), $c(1/2) \simeq -0.0004$. As a result, the finite part of $\zeta_{\mathcal{M}}^{\mathrm{I},\mathcal{D}}(s)$ is

$$p(1/2) + c(1/2) \simeq 0.0093 - 0.0004 = 0.0089 \tag{B.8}$$

in good agreement with the finite part of (3.32).

C Appendix: 'naïve' zeta-function regularization

Let's calculate again the e.m. case in $D = 2$, but making now the addition of internal and external modes before the angular momentum summation. After adding up (3.35) to (4.11) for $\alpha = 0$ (i.e. purely Neumann b.c.) and rescaling $x \to \nu x$, we find

$$\begin{aligned} \zeta_\nu^{\mathcal{N}}(s) &\equiv \left[\zeta_\nu^{\mathrm{I},\mathcal{R}}(s) + \zeta_\nu^{\mathrm{II},\mathcal{R}}(s)\right]_{\alpha=0} \\ &= \frac{s}{\pi} \sin\frac{\pi s}{2}\, \nu^{-s} \int_0^\infty dx\, x^{-s-1} \ln\left[-2\,\nu x I_\nu'(\nu x) K_\nu'(\nu x)\right]. \end{aligned} \tag{C.1}$$

Making use of the u.a.e.'s for $I_\nu'(\nu x)$ and $K_\nu'(\nu x)$, taking advantage of previous steps and of parity reasoning,

$$\ln\left[-2\nu x I_\nu'(\nu x) K_\nu'(\nu x)\right] \sim \ln\frac{1}{x t(x)} + \sum_{n\geq 1} \frac{2\,\mathcal{U}_{2n}^{\mathrm{I},\mathcal{R}}(t(x))\Big|_{\alpha=0}}{\nu^{2n}} \tag{C.2}$$

where $t(x)$ is the one in (3.14) and the $\mathcal{U}_{2n}^{\mathrm{I},\mathcal{R}}$'s are given in (3.37). Subtracting up to $n = N$ terms from the integrand, one may write

$$\zeta_\nu^{\mathcal{N}}(s) = \frac{s}{\pi} \sin\frac{\pi s}{2}\, \nu^{-s} \left[\int_0^\infty dx\, x^{-s-1} \ln\frac{(1+x^2)^{1/2}}{x} + \sum_{n=1}^{N} \frac{2\,\mathcal{J}_{2n}^{\mathrm{I},\mathcal{R}}(s)\Big|_{\alpha=0}}{\nu^{2n}} + \mathcal{S}_N^{\mathcal{N}}(s;\nu)\right], \tag{C.3}$$

By (3.11), we know that the above integral amounts to $2 \cdot \frac{\pi}{4s \sin\frac{\pi s}{2}}$ The $\mathcal{J}_{2n}^{\mathrm{I},\mathcal{R}}$'s are already specified by the integrals (3.40). $\mathcal{S}_N^{\mathcal{N}}(s;\nu)$ denotes the corresponding residual integral, in the style of (3.39), containing the difference between the exact integrand and its N-term approximation.

Taking into account the $D = 2$ degeneracies and noting that $\nu(2,l) = l$, the complete zeta function is put as

$$\zeta_{\mathcal{M}}^{\mathcal{N}}(s) = a^s \left[\zeta_0^{\mathcal{N}}(s) + 2\sum_{l=1}^{\infty} \zeta_l^{\mathcal{N}}(s)\right]. \tag{C.4}$$

The $l = 0$ contribution to the Casimir energy is therefore

$$\frac{1}{2a}\zeta_0^{\mathcal{N}}(-1) = \frac{1}{2a}\left[\zeta_0^{\mathrm{I},\mathcal{R}}(s) + \zeta_0^{\mathrm{II},\mathcal{R}}(s)\right]_{\alpha=0,\ s=-1} = -\frac{1}{a}0.25352 \tag{C.5}$$

(formula (4.16)). This agrees with (3.5) in ref. [37]. On the other hand, the set of $l \neq 0$ modes yields a contribution which amounts to the $s \to -1$ limit of the following quantity:

$$\frac{1}{a}\sum_{l=1}^{\infty}\zeta_l^{\mathcal{N}}(s) = \frac{1}{a}\left[\frac{1}{2}\zeta_R(s) + \frac{s}{\pi}\sin\frac{\pi s}{2}\left\{\sum_{n=1}^{N} 2\,\zeta_R(2n+s)\,\mathcal{J}_{2n}^{\mathrm{I},\mathcal{R}}(s)\Big|_{\alpha=0} + \sum_{l=1}^{\infty}\mathcal{S}_N^{\mathcal{N}}(s;l)\,l^{-s}\right\}\right]. \tag{C.6}$$

At $s=-1$ the l-sum causes no problem, as it is finite and relatively small. However, the n-sum contains a divergence in its $n=1$-term, coming from the pole of ζ_R when its argument equals one. Bearing this in mind, we find the residue of (C.6) at $s=-1$ to be $\frac{2}{\pi}\mathcal{J}_2^{\mathrm{I},\mathcal{R}}(-1)\Big|_{\alpha=0} = -\frac{5}{128}$, as given by (5.5) (up this point, we have just been doing the same calculation but re-grouping things in a convenient way). Now, if we deliberately ignore this singularity and throw away as 'corrections' the $l-$ and $n-$ sums in (C.6), we simply get

$$\frac{1}{2a}\zeta_R(-1) = -\frac{1}{24a}, \tag{C.7}$$

which is precisely the 'LT' ('leading term') contribution of the $l \neq 0$ modes to the Casimir effect given in eq. (3.9) of ref. [37], i.e. it is the part which comes from keeping just the first term on the r.h.s. of (C.2). Of course, viewed from inside our zeta-regularization context, such an approximation in unjustifiable, since the whole value has a divergence attached which we dare not call 'spurious'.

Doing the same with the scalar Dirichlet modes, one easily sees that the $l=0$ part is one half of (4.10), in agreement with result (A13) of ref. [37], and the $l \neq 0$ piece analogous to the above mentioned approximation is

$$-\frac{1}{2a}\zeta_R(-1) = \frac{1}{24a}, \tag{C.8}$$

which amounts to the 'LT' result (A16) in ref. [37] and whose validity rests on the same flimsy assumptions.

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References

[1] H.B.G. Casimir, *Proc. Kon. Ned. Akad. Wetenschap* **51** (1948) 793.

[2] D.J. Toms, *Phys. Rev.* **D 21** (1980) 928; *Phys. Rev.* **D 21** (1980) 2805.

[3] J. Ambjørn and S. Wolfram, *Ann. Phys.* **147** (1983) 1.

[4] G. Plunien, B. Müller and W. Greiner, *Phys. Rep.* **134** (1986) 87.

[5] L.H. Ford, *Phys. Rev.* **D36** (1988) 528.

[6] S.K. Blau, M. Visser and A. Wipf, *Nucl. Phys.* **B310** (1988) 163.

[7] I. Brevik and G.H. Nyland, *Ann. Phys.* **230** (1994) 321;
I. Brevik, H. Skurdal and R. Sollie, *J. Phys.* **A 27** (1994) 6853.

[8] E. Robaschik and E. Wieczorek, *Ann. Phys.* **236** (1994) 43.

[9] C. Erbelein, *J. Phys.* **A 25** (1992) 3015, 3039;
P.A. Maïa Neto and S. Reynaud, *Phys. Rev.* **A 47** (1993) 1639;
Y. Pomeau, *Europhys. Lett.* **27** (1994) 377.

[10] C.M. Bender and K.A. Milton, *Phys. Rev.* **D 50** (1994) 6547.

[11] A.V. Nesteruk, *Europhys. Lett.* **32** (1995) 455.

[12] D. Deutsch and P. Candelas, *Phys. Rev.* **D 20** (1979) 3063.

[13] K.A. Milton, L.L. DeRaad Jr., and J. Schwinger, *Ann. Phys.* (N.Y.) **115** (1978) 388.

[14] R. Balian and B. Duplantier, *Ann. Phys.* (N.Y.) **112** (1978) 165.

[15] H.P. McKean Jr. and I.M. Singer, *J. Diff. Geom.* **1** (1967) 43;
R. Seeley, *Amer. J. Math.* **91** (1969) 889;
P.B. Gilkey, *Proc. Symp. Pure Math.* **27** (1975) 265; *J. Diff. Geom.* **10** (1975) 601; *Adv. in Math.* **102** (1993) 129;
T.P. Branson and P.B. Gilkey, *Commun. Partial Diff. Eq.* **15** (1990) 245;
B.S. De Witt, *Phys. Rep.* **19** (1975) 295.

[16] G. Kennedy, *J. Phys.* **A 11** (1978) L173.

[17] B. Durhuus, P. Olesen and J.L. Petersen, *Nucl. Phys.* **B 198** (1982) 157.

[18] M. Bordag, E. Elizalde and K. Kirsten, hep-th/9503023, *J. Math. Phys.* accepted.

[19] J.S. Dowker, hep-th/9506042; hep-th/9508082, *Phys. Lett.* **B** accepted.

[20] K.A. Milton, *Phys. Rev.* **D 22** (1980) 1441; *Phys. Rev.* **D 27** (1983) 439;
Ann. Phys. (N.Y.) **150** (1983) 432.

[21] L.S. Brown and G.J. MacLay, *Phys. Rev.* **184** (1969) 1272;
J.S. Dowker and R. Critchley, *Phys. Rev.* **D 13** (1976) 3224.

[22] S.W. Hawking, *Commun. Math. Phys.* **55** (1978) 133.

[23] M.V. Berry and M. Robnik, *J. Phys.* **A 19** (1986) 649;
M.V. Berry, *J. Phys.* **A 19** (1986) 2281; *J. Phys.* **A 20** (1987) 2389.

[24] C. Itzykson, P. Moussa and J.M. Luck, *J. Phys.* **A 19**, L111 (1986);
R. M. Ziff, *J. Phys.* **A 19** (1986) 3923.

[25] J.S. Dowker, *J. Phys.* **A 11** (1978) 2255; *Phys. Rev.* **D 29** (1984) 2773.

[26] P. Candelas and S. Weinberg, *Nucl. Phys.* **B237** (1984) 397;
A. Chodos and E. Myers, *Ann. Phys.* **156** (1984) 412; *Phys. Rev.* **D 31** (1985) 3064;
R. Kantowski and K.A. Milton, *Phys. Rev.* **D 35** (1987) 549; *Phys. Rev.* **D 36** (1987) 3712.

[27] G. Ghika and M. Vişinescu, *Nuovo Cimento* **46 A** (1978) 25.

[28] T. Kunimasa and K. Uehara, *Nucl. Phys.* **B279** (1987) 608.

[29] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, *Zeta Regularization Techniques with Applications*, World Scientific (1994).

[30] E. Elizalde, S. Leseduarte and A. Romeo, *J. Phys.* **A 26** (1993) 2409.

[31] S. Leseduarte and A. Romeo, *J. Phys.* **A 27** (1994) 2483.

[32] A. Romeo, *Scalar Casimir effect in a circular Aharonov-Bohm quantum billiard*, unpublished;
S. Leseduarte and A. Romeo, *Influence of an external magnetic flux on the zero-point energy of a field subject to spherical boundary conditions*, in preparation.

[33] G.N. Watson, *A treatise on the Theory of Bessel Functions*, 2nd edition, Cambridge University Press, Cambridge (1944);
N. Kishore, *Proc. Amer. Math. Soc.* **14** (1963) 527;
E.C. Obi, *J. Math. Anal. Appl.* **52** (1975) 648;
J. Hawkins, *On a zeta function associated with Bessel's equation*, PhD Thesis, University of Illinois (1983);
K.B. Stolarsky, *Mathematika* **32** (1985) 96.

[34] F. Steiner, *Fortschr. Phys.* **35** (1987) 87.

[35] Y-H. Chen, F. Wilczek, E. Witten and B.I. Halperin, *Int. J. Mod. Phys.* **B3** (1989) 1001;
S. Deser, *Phys. Rev. Lett.* **64** (1990) 611;
L. Alberto, *Anyons*, Springer Verlag (1992).

[36] K.A. Milton and Y.J. Ng, *Phys. Rev.* **D 42** (1990) 2875.

[37] K.A. Milton and Y.J. Ng, *Phys. Rev.* **D 46** (1992) 842.

[38] A. Romeo, *Phys. Rev.* **D 52** (1995) 7308.

[39] A.O. Barvinsky, A. Yu. Kamenshchik and I.P. Karmazin, *Ann. Phys.* **219** (1992) 201.

[40] J.C. Slater and N.H. Frank, *Electromagnestism*, Dover, New York (1969).

[41] T.H. Boyer, *Phys. Rev.* **174** (1968) 174.

[42] C.M. Bender, S. Boettcher and L. Lipatov, *Phys. Rev. Lett.* **168** (1992) 3764; *Phys. Rev.* **D 146** (1992) 5557;
C.M. Bender and S. Boettcher, *J. Math. Phys.* **135** (1994) 1914; *Phys. Rev.* **D 148** (1993) 4919.

[43] A.A. Kvitsinsky, *J. Phys.* **A 28** (1995) 1753; *J. Math. Anal. Appl.*, **196** (1995) 947.

[44] N. Ja. Vilenkin, *Fonctions spéciales et théorie de la répresentation des groups*, Dunod, Paris (1969).

[45] A. Cappelli and A. Coste, *Nucl. Phys.* **B314** (1989) 707.

[46] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1972).

[47] C.M. Bender and P. Hays, *Phys. Rev.* **D 14** (1976) 2622.

[48] P. Candelas, *Ann. Phys.* (N.Y.) **143** (1982) 241.

[49] T. Muta, *Foundations of Quantum Chromodynamics*, World Scientific (1987).

[50] W. Lukosz, *Physica* **56** (1971) 109.

[51] S. Sen, *J. Math. Phys.* **22** (1981) 2968; **25** (1984) 2000.

[52] E. Elizalde and A. Romeo, *Int. J. Mod. Phys.* **A** (1990) 1653.

[53] G. Cognola, L. Vanzo and S. Zerbini, *J. Math. Phys.* **33** (1992) 222.

[54] L.L. De Raad and K.A. Milton, *Ann. Phys.* **136** (1981) 229.